PART - A
Unit - II

If an expression $F(x)$ at $x = a$ assumes forms like $0/0$, ∞/∞ , $0 \times \infty$, $\infty - \infty$, 0^0 , ∞^0 , 1^∞ which do not represent any value are called *indeterminate forms*. The concept of limit gives a meaningful value for the function $F(x)$ at $x = a$ overcoming these indeterminate forms.

The reader is familiar with the evaluation of limit mostly in the cases of $0/0$ or ∞/∞ without the involvement of differentiation. Few more indeterminate forms: $\infty - \infty$, $\infty \times 0$, ∞^0 , 0^0 ,1^∞ can be reduced to the two basic indeterminate forms $0/0$ and ∞/∞. Then limit is found passing through a process of differentiation warranted by a very simple rule called L' Hospital's (*French Mathematician*) rule which is established by using Cauchy's mean value theorem.

Statement : If $f(x)$ and $g(x)$ are two functions such that

(i) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ *ie.,* $f(a) = 0 = g(a)$

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note : ***Extension of the theorem***

If $f'(a) = 0$ and $g'(a) = 0$ then we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} \text{ and so on.}$$

Working procedure for problems by applying L' Hospital's rule

- The rule is applicable for the form 0/0. It can also be applied for the form ∞/∞ as we can write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left\{ \frac{1/g(x)}{1/f(x)} \right\}$$

where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$

- However while applying the rule in this case also, we follow the usual procedure of differentiating the numerator $f(x)$ and the denominator $g(x)$ separately. If the indeterminate form persists after applying the rule once, we can apply the rule repeatedly till we arrive at a definite value. It is highly advisible to look for simplification at each stage. Problems have been bifurcated into four types and the procedure too has been explained separately in each type.

- The following four standard limits and well known simple properties connected with limits can be readily used.

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x \to 0} \frac{x}{\sin x} = 1$ (iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ (iv) $\lim_{x \to 0} \frac{x}{\tan x} = 1$

Type-1

The rule can be applied directly in the case of forms 0/0 and ∞/∞. In the cases of $\infty - \infty$ and $\infty \times 0$, we have to employ simple methods (*taking L.C.M, using equivalent trigonometric expressions etc.*) to simplify the given expression in bringing it to the form 0/0 or ∞/∞ so that the L' Hospital's rule can be employed.

WORKED PROBLEMS

Evaluate the following limits

1. $\lim_{x \to 0} \frac{x e^x - \log(1+x)}{x^2}$

2. $\lim_{x \to 0} \frac{1 - \cos x}{x \log(1+x)}$

3. $\lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2}$

4. $\lim_{x \to 0} \frac{a^x - b^x}{x}$

5. $\lim_{x \to 0} \frac{\sinh x - x}{\sin x - x \cos x}$

6. $\lim_{x \to a} \frac{x^x - a^x}{x^a - a^a}$

7. $\lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x}$

8. $\lim_{x \to 0} \log_{\sin x} \sin 2x$

9. $\lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x}$

10. $\lim_{x \to 0} \log_{\tan bx} \tan ax$

1. Let $k = \lim\limits_{x \to 0} \dfrac{x e^x - \log(1+x)}{x^2} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospitals rule,

$$k = \lim_{x \to 0} \frac{x e^x + e^x - 1/1+x}{2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{x e^x + e^x + e^x + 1/(1+x)^2}{2} = \frac{0+1+1+1}{2} = \frac{3}{2}$$

Thus $\quad$ **$k = 3/2$**

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2. Let $k = \lim\limits_{x \to 0} \dfrac{1 - \cos x}{x \log(1+x)} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$= \lim_{x \to 0} \frac{\sin x}{x \cdot (1/1 + x) + \log(1+x)} \quad \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x}{x \cdot -1/(1+x)^2 + 1/1+x + 1/1+x} = \frac{1}{0+1+1} = \frac{1}{2}$$

Thus $\quad$ **$k = 1/2$**

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3. Let $k = \lim\limits_{x \to \pi/2} \dfrac{\log(\sin x)}{(\pi/2 - x)^2} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule, $k = \lim\limits_{x \to \pi/2} \dfrac{\cos x/\sin x}{-2(\pi/2 - x)}$

ie., $\quad = \lim\limits_{x \to \pi/2} \dfrac{\cot x}{-2(\pi/2 - x)} \quad \ldots \left(\dfrac{0}{0}\right)$

Now $\quad k = \lim\limits_{x \to \pi/2} \dfrac{-\operatorname{cosec}^2 x}{2} = \dfrac{-1}{2}$

Thus $\quad$ **$k = -1/2$**

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4. Let $k = \lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1} = \log a - \log b = \log(a/b)$$

Thus $\quad$ **$k = \log(a/b)$**

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5. Let $k = \lim\limits_{x \to 0} \dfrac{\sin h\, x - x}{\sin x - x \cos x} \cdots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{\cos h\, x - 1}{\cos x + x \sin x - \cos x} = \lim_{x \to 0} \frac{\cos h\, x - 1}{x \sin x} \cdots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to 0} \frac{\sin h\, x}{x \cos x + \sin x} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos h\, x}{-x \sin x + \cos x + \cos x} = \frac{1}{2}$$

Thus $k = 1/2$

(**Note** : *We have applied the rule thrice in this example*)

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6. Let $k = \lim\limits_{x \to a} \dfrac{x^x - a^x}{x^a - a^a} \cdots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to a} \frac{x^x (1 + \log x) - a^x \log a}{a x^{a-1} - 0} \cdots \left(\frac{0}{0}\right)$$

$$k = \frac{a^a (1 + \log a) - a^a \log a}{a \cdot a^{a-1}} = \frac{a^a}{a^a} = 1$$

Thus $k = 1$

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7. Let $k = \lim\limits_{x \to \pi/2} \dfrac{\log (x - \pi/2)}{\tan x} \cdots \left(\dfrac{-\infty}{\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to \pi/2} \frac{1/(x - \pi/2)}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\cos^2 x}{(x - \pi/2)} \cdots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to \pi/2} \frac{-2 \cos x \sin x}{1} = 0$$

Thus $k = 0$

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8. Let $k = \lim_{x \to 0} \log_{\sin x} \sin 2x$

(*We have the property* $\log_b a = \log a / \log b$ *for any base*)

Now $k = \lim_{x \to 0} \dfrac{\log(\sin 2x)}{\log(\sin x)} \quad \ldots \left(\dfrac{-\infty}{-\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2\cos 2x/\sin 2x}{\cos x/\sin x} = \lim_{x \to 0} \frac{2\cot 2x}{\cot x} = \lim_{x \to 0} \frac{2\tan x}{\tan 2x} \quad \ldots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to 0} \frac{2\sec^2 x}{2\sec^2 2x} = 1.$$

Thus $\mathbf{k = 1}$

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9. Let $k = \lim_{x \to 0} \dfrac{\log x}{\operatorname{cosec} x} \quad \ldots \left(\dfrac{-\infty}{\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x} = -\lim_{x \to 0} \frac{\sin x}{x} \cdot \tan x$$

$$= -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \tan x = -1 \cdot 0 = 0$$

Thus $\mathbf{k = 0}$

Remark : $\sin x \tan x/x$ *is of the form* $0/0$ *and the rule could have been applied again to obtain the answer. But it is always advisible to look for simplification and use standard limits at the right juncture.*

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10. Let $k = \lim_{x \to 0} \log_{\tan bx} \tan ax = \lim_{x \to 0} \dfrac{\log(\tan ax)}{\log(\tan bx)} \quad \ldots \left(\dfrac{-\infty}{-\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{a\sec^2 ax/\tan ax}{b\sec^2 bx/\tan bx} = \lim_{x \to 0} \frac{a\sec^2 ax \tan bx}{b\sec^2 bx \tan ax}$$

ie.,

$$= \frac{a}{b} \lim_{x \to 0} \frac{1}{\cos^2 ax} \cdot \frac{\cos ax}{\sin ax} \cdot \frac{\sin bx}{\cos bx} \cdot \cos^2 bx$$

$$= \frac{a}{b} \lim_{x \to 0} \frac{\sin bx \cos bx}{\sin ax \cos ax} = \frac{a}{b} \lim_{x \to 0} \frac{\sin 2bx}{\sin 2ax} \quad \ldots \left(\frac{0}{0}\right)$$

$\therefore \qquad k = \dfrac{a}{b} \lim\limits_{x \to 0} \dfrac{2b \cos 2\,bx}{2a \cos 2\,ax} = 1$

Thus $\quad \mathbf{k = 1}$

Note : *Alternative simplification after the first step*

$$k = \lim_{x \to 0} \frac{a \sec^2 ax}{b \sec^2 bx} \cdot \lim_{x \to 0} \frac{\tan bx}{\tan ax}$$

$$= \frac{a}{b} \lim_{x \to 0} \frac{\tan bx}{\tan ax} \cdots \left(\frac{0}{0}\right) \quad \text{since} \quad \sec 0 = 1$$

Now, applying the rule again we get

$$k = \frac{a}{b} \lim_{x \to 0} \frac{b \sec^2 bx}{a \sec^2 ax} = 1$$

Thus $\quad \mathbf{k = 1}$

Evaluate the following limits

11. $\lim\limits_{x \to 1} \left[\dfrac{x}{x-1} - \dfrac{1}{\log x} \right]$

12. $\lim\limits_{x \to 2} \left[\dfrac{1}{x-2} - \dfrac{1}{\log (x-1)} \right]$

13. $\lim\limits_{x \to 0} \left[\dfrac{a}{x} - \cot (x/a) \right]$

14. $\lim\limits_{x \to 0} \left[\dfrac{1}{x} - \dfrac{\log (1+x)}{x^2} \right]$

15. $\lim\limits_{x \to \pi/2} [\, 2x \tan x - \pi \sec x \,]$

16. $\lim\limits_{x \to 0} \tan x \cdot \log x$

17. $\lim\limits_{x \to \infty} x \tan (1/x)$

18. $\lim\limits_{x \to 1} (1 - x^2) \tan (\pi x/2)$

19. $\lim\limits_{x \to 0} x \log x$

20. $\lim\limits_{x \to a} \log [\, 2 - (x/a) \,] \cot (x - a)$

11. Let $k = \lim\limits_{x \to 1} \left[\dfrac{x}{x-1} - \dfrac{1}{\log x} \right] \cdots (\infty - \infty)$

We need to simplify the given expression.

ie., $\qquad = \lim\limits_{x \to 1} \left[\dfrac{x \log x - (x-1)}{(x-1) \log x} \right] \cdots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 1} \left[\frac{x \cdot 1/x + \log x - 1}{(x-1) \cdot 1/x + \log x} \right] = \lim_{x \to 1} \left[\frac{x \log x}{(x-1) + x \log x} \right] \cdots \left(\frac{0}{0} \right)$$

$$\therefore \quad k = \lim_{x \to 1} \frac{1 + \log x}{1 + (1 + \log x)} = \frac{1+0}{2+0} = \frac{1}{2}$$

Thus **$k = 1/2$**

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12. $$k = \lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)} \right] \cdots (\infty - \infty)$$

ie., $$= \lim_{x \to 2} \left[\frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)} \right] \cdots \left(\frac{0}{0} \right)$$

Applying L' Hospital's rule,

$$= \lim_{x \to 2} \left[\frac{(1/x-1) - 1}{(x-2/x-1) + \log(x-1)} \right]$$. We shall simplify.

ie., $$= \lim_{x \to 2} \left[\frac{2-x}{(x-2) + (x-1)\log(x-1)} \right] \cdots \left(\frac{0}{0} \right)$$

$$\therefore \quad k = \lim_{x \to 2} \left[\frac{-1}{1 + 1 + \log(x-1)} \right] = \frac{-1}{2}$$

Thus **$k = -1/2$**

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13. Let $$k = \lim_{x \to 0} \left[\frac{a}{x} - \cot(x/a) \right] \cdots (\infty - \infty)$$

ie., $$= \lim_{x \to 0} \left[\frac{a}{x} - \frac{\cos(x/a)}{\sin(x/a)} \right]$$

$$= \lim_{x \to 0} \left[\frac{a \sin(x/a) - x \cos(x/a)}{x \sin(x/a)} \right] \cdots \left(\frac{0}{0} \right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \left[\frac{a \cdot 1/a \cdot \cos(x/a) + x \cdot 1/a \cdot \sin(x/a) - \cos(x/a)}{x \cdot 1/a \cdot \cos(x/a) + \sin(x/a)} \right]$$

ie., $$= \lim_{x \to 0} \left[\frac{x \sin(x/a)}{x \cos(x/a) + a \sin(x/a)} \right] \cdots \left(\frac{0}{0} \right)$$

$$\therefore \quad k = \lim_{x \to 0} \left[\frac{x \cdot 1/a \cos(x/a) + \sin(x/a)}{x \cdot -1/a \cdot \sin(x/a) + \cos(x/a) + \cos(x/a)} \right]$$

ie., $= \dfrac{0+0}{0+1+1} = \dfrac{0}{2} = 0$

Thus $\boldsymbol{k = 0}$

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14. Let $k = \lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{\log(1+x)}{x^2} \right] \ldots \left(\infty - \dfrac{0}{0} \right)$

ie., $= \lim_{x \to 0} \dfrac{x - \log(1+x)}{x^2} \ldots \left(\dfrac{0}{0} \right)$

Applying L'Hospital's rule,

$$k = \lim_{x \to 0} \frac{1 - (1/1+x)}{2x} \ldots \left(\frac{0}{0} \right)$$

$$\therefore \quad k = \lim_{x \to 0} \frac{1/(1+x)^2}{2} = \frac{1}{2}$$

Thus $\boldsymbol{k = 1/2}$

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15. Let $k = \lim_{x \to \pi/2} (2x \tan x - \pi \sec x) \ldots (\infty - \infty)$

ie., $= \lim_{x \to \pi/2} \left[2x \cdot \dfrac{\sin x}{\cos x} - \pi \cdot \dfrac{1}{\cos x} \right]$

ie., $= \lim_{x \to \pi/2} \left[\dfrac{2x \sin x - \pi}{\cos x} \right] \ldots \left(\dfrac{0}{0} \right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to \pi/2} \left[\frac{2x \cos x + 2 \sin x}{-\sin x} \right] = -2$$

Thus $\boldsymbol{k = -2}$

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16. Let $k = \lim_{x \to 0} \tan x \cdot \log x \ldots (0 \times -\infty)$

ie., $= \lim_{x \to 0} \dfrac{\log x}{\cot x} \ldots \left(-\dfrac{\infty}{\infty} \right)$

Now applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x} = -\lim_{x \to 0} \frac{\sin^2 x}{x}$$

ie., $$= -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \sin x = -1 \cdot 0 = 0$$

Thus $\mathbf{k = 0}$

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17. Let $k = \lim_{x \to \infty} x \tan(1/x) \ldots (\infty \times 0)$

ie., $$= \lim_{x \to \infty} \frac{\tan(1/x)}{(1/x)} \quad \text{Put } 1/x = y \text{ ; } y \to 0 \text{ as } x \to \infty$$

Hence $k = \lim_{y \to 0} \frac{\tan y}{y} = 1$

Thus $\mathbf{k = 1}$

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18. Let $k = \lim_{x \to 1} (1 - x^2) \tan(\pi x/2) \ldots (0 \times \infty)$

ie., $$= \lim_{x \to 1} \frac{(1 - x^2)}{\cot(\pi x/2)} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 1} \frac{-2x}{-\pi/2 \operatorname{cosec}^2(\pi x/2)} = \frac{4}{\pi}$$

Thus $\mathbf{k = 4/\pi}$

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19. Let $k = \lim_{x \to 0} x \log x \ldots (0 \times -\infty)$

ie., $$= \lim_{x \to 0} \frac{\log x}{(1/x)} \ldots \left(\frac{-\infty}{\infty}\right)$$

Now applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} -x = 0$$

Thus $\mathbf{k = 0}$

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20. Let $k = \lim\limits_{x \to a} \log[2-(x/a)] \cot(x-a) \ldots (0 \times \infty)$

ie., $$= \lim_{x \to a} \frac{\log[2-(x/a)]}{\tan(x-a)} \cdots \left(\frac{0}{0}\right)$$

Now applying L' Hospital's rule,

$$k = \lim_{x \to a} \frac{1}{2-(x/a)} \cdot \frac{-1}{a} \cdot \frac{1}{\sec^2(x-a)} = -\frac{1}{a}$$

Thus $k = -1/a$

Type-2

The given expression or its simplified form will be in the 0/0 form when $x = 0$ or as $x \to 0$ but will involve terms of the form $x^2 \sin x$, $x \sin^3 x$, $x \tan^2 x$ etc. In the event of applying the rule, the differentiation becomes tedious and we should not venture to do so. We can conveniently modify such terms so as to involve $(\sin x/x)^k$ or $(\tan x/x)^k$ or $(x/\sin x)^k$ or $(x/\tan x)^k$ which can be separated out from the given expression. These terms become 1 as $x \to 0$ with the result we will be left with a simple expression (*product gets eliminated*) in the 0/0 form for the application of L' Hospital's rule. Simplification at each step has to be explored.

Evaluate the following limits

21. $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$

22. $\lim\limits_{x \to 0} \dfrac{x^2 + 2\cos x - 2}{x \sin^3 x}$

23. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

24. $\lim\limits_{x \to 0} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

25. $\lim\limits_{x \to 0} \left[\dfrac{1}{x^2} - \cot^2 x\right]$

26. $\lim\limits_{x \to 0} \left[\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right]$

21. Let $k = \lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x} \cdots \left(\dfrac{0}{0}\right)$

ie., $$= \lim_{x \to 0} \frac{\tan x - x}{x^2 \cdot \frac{\tan x}{x} \cdot x} = \lim_{x \to 0} \frac{\tan x - x}{x^3} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)$$

Hence $k = \lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3} \cdot 1 \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{\tan^2 x}{3x^2}$$

ie., $$= \frac{1}{3} \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2 = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

Thus $\boldsymbol{k = 1/3}$

22. Let $k = \lim\limits_{x\to 0} \dfrac{x^2 + 2\cos x - 2}{x\sin^3 x} \ldots \left(\dfrac{0}{0}\right)$

ie., $$= \lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x \cdot \frac{\sin^3 x}{x^3} \cdot x^3} = \lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x^4} \cdot \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^3$$

Hence $k = \lim\limits_{x\to 0} \dfrac{x^2 + 2\cos x - 2}{x^4} \cdot 1 \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x\to 0} \frac{2x - 2\sin x}{4x^3} \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2 - 2\cos x}{12x^2} \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\sin x}{24x} = \frac{1}{12} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{12}\cdot 1 = \frac{1}{12}$$

Thus $\boldsymbol{k = 1/12}$

23. Let $k = \lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin x} \ldots \left(\dfrac{0}{0}\right)$

ie., $$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \cdot \frac{\sin x}{x} \cdot x}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \lim_{x\to 0} \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot 1 \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{e^x + e^{-x} - (2/1+x)}{2x} \cdots \left(\frac{0}{0}\right)$$

$$k = \lim_{x \to 0} \frac{e^x - e^{-x} + 2/(1+x)^2}{2} = 1$$

Thus $k = 1$

24. Let $k = \lim_{x \to 0} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x} \cdots \left(\dfrac{0}{0}\right)$

ie., $$= \lim_{x \to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x \to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \cdot 1 \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospitals rule,

$$k = \lim_{x \to 0} \frac{\cos x + \sin x - 1/1-x}{3x^2} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x + \cos x - 1/(1-x)^2}{6x} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\cos x - \sin x - 2/(1-x)^3}{6} = \frac{-3}{6} = \frac{-1}{2}$$

Thus $k = -1/2$

25. Let $k = \lim_{x \to 0} \left[\dfrac{1}{x^2} - \cot^2 x\right] \cdots (\infty - \infty)$

ie., $$= \lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\tan^2 x}\right] = \lim_{x \to 0} \left[\frac{\tan^2 x - x^2}{x^2 \tan^2 x}\right]$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \cdot 1 \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2 \tan x \sec^2 x - 2x}{4x^3} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \tan x \cdot 2 \sec^2 x \tan x + 2 \sec^4 x - 2}{12x^2}$$

ie., $$= 2 \lim_{x \to 0} \frac{2 \sec^2 x \tan^2 x + \sec^4 x - 1}{12x^2} \cdots \left(\frac{0}{0}\right)$$

(*Further differentiation will be tedious and hence we simplify the term* $\sec^4 x - 1$)

Now, $(\sec^4 x - 1) = (\sec^2 x - 1)\ (\sec^2 x + 1)$

ie., $(\sec^4 x - 1) = \tan^2 x(\sec^2 x + 1) = \sec^2 x \tan^2 x + \tan^2 x$

Hence $$k = \lim_{x \to 0} \frac{2 \sec^2 x \tan^2 x + \sec^2 x \tan^2 x + \tan^2 x}{6x^2}$$

$$= \lim_{x \to 0} \frac{\tan^2 x (3 \sec^2 x + 1)}{6x^2}$$

$$= \frac{1}{6} \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2 \cdot \lim_{x \to 0} (3 \sec^2 x + 1)$$

$$= \frac{1}{6} \cdot 1 \cdot 4 = \frac{2}{3}$$

Thus $k = 2/3$

26. Let $$k = \lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] \cdots (\infty - \infty)$$

ie., $$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4} \cdot 1 \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2 \sin x \cos x - 2x}{4x^3} = \lim_{x \to 0} \frac{\sin 2x - 2x}{4x^3} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2}{12x^2} \quad \text{or} \quad \lim_{x \to 0} \frac{\cos 2x - 1}{6x^2}$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{-2\sin^2 x}{x^2} = \frac{-1}{3} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = \frac{-1}{3} \cdot 1 = \frac{-1}{3}$$

Thus $k = -1/3$

Indeterminate forms : 1^{∞}

It is evident that the function involved will be of the form $[f(x)]^{g(x)}$ and we have to find the limit as $x \to a$.

Let $k = \lim_{x \to a} [f(x)]^{g(x)}$

Taking logarithms on both sides we have,

$$\log_e k = \lim_{x \to a} g(x) \cdot \log[f(x)]$$

We can evaluate the limit on the R.H.S as already discussed and let us suppose that the limit is equal to l.

ie., $\log_e k = l \Rightarrow k = e^l$ which is the required limit.

Remark : *One of the common question is that why 1^{∞} is indeterminate ?*

Let $k = \lim_{x \to a} [f(x)]^{g(x)} \ldots 1^{\infty}$

$\Rightarrow \quad \log_e k = \lim_{x \to a} g(x) \log f(x) \ldots \infty \times \log 1 = \infty \times 0$

which is indeterminate. On the otherhand if $k = \lim_{x \to a} [f(x)]^{g(x)}$ is of the form c^{∞} where $c \neq 1$ we have

$$\log_e k = \lim_{x \to a} g(x) \log f(x) = \infty \times \log c = \infty$$

Evaluate the following limits

27. $\lim_{x \to 1} x^{1/1-x}$

28. $\lim_{x \to 0} (\cos x)^{1/x^2}$

29. $\lim_{x \to \pi/2} (\sin x)^{\tan x}$

30. $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$

27. Let $k = \lim_{x \to 1} x^{1/1-x} \ldots (1^{\infty})$

$\Rightarrow \quad \log_e k = \lim_{x \to 1} \frac{1}{1-x} \log x = \lim_{x \to 1} \frac{\log x}{1-x} \ldots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 1} \frac{1/x}{-1} = -1$$

ie., $\quad \log_e k = -1$

Thus $\quad \boldsymbol{k = e^{-1} = 1/e}$

---

28. Let $k = \lim_{x \to 0} (\cos x)^{1/x^2} \ldots (1^{\infty})$

$\Rightarrow \quad \log_e k = \lim_{x \to 0} \frac{\log(\cos x)}{x^2} \ldots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{-\sin x/\cos x}{2x} = \frac{-1}{2} \lim_{x \to 0} \frac{\tan x}{x} = \frac{-1}{2} \cdot 1$$

ie., $\quad \log_e k = -1/2 \Rightarrow k = e^{-1/2} = 1/\sqrt{e}$

Thus $\quad \boldsymbol{k = 1/\sqrt{e}}$

---

29. Let $k = \lim_{x \to \pi/2} (\sin x)^{\tan x} \ldots (1^\infty)$

$$\Rightarrow \quad \log_e k = \lim_{x \to \pi/2} \tan x \log(\sin x) \ldots (\infty \times 0)$$

ie.,
$$= \lim_{x \to \pi/2} \frac{\log(\sin x)}{\cot x} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to \pi/2} \frac{\cos x/\sin x}{-\operatorname{cosec}^2 x} = \lim_{x \to \pi/2} -\sin x \cos x = 0$$

ie., $\log_e k = 0$

Thus $\mathbf{k = e^0 = 1}$

30. Let $k = \lim_{x \to a} \left[2 - (x/a)\right]^{\tan(\pi x/2a)} \ldots (1^\infty)$

$$\Rightarrow \quad \log_e k = \lim_{x \to a} \tan(\pi x/2a) \cdot \log\left[2 - (x/a)\right] \ldots (\infty \times 0)$$

ie.,
$$= \lim_{x \to a} \frac{\log\left[2 - (x/a)\right]}{\cot(\pi x/2a)} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to a} \frac{\dfrac{1}{\left[2 - (x/a)\right]} \times \dfrac{-1}{a}}{-\operatorname{cosec}^2(\pi x/2a) \times \pi/2a} = \frac{2}{\pi}$$

ie., $\log_e k = 2/\pi$

Thus $\mathbf{k = e^{2/\pi}}$

31. Let $k = \lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x} \ldots (1^\infty)$

$$\Rightarrow \quad \log_e k = \lim_{x \to 0} \frac{\log\{(a^x + b^x)/2\}}{x} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{\frac{2}{a^x + b^x} \cdot \frac{1}{2}(a^x \log a + b^x \log b)}{1}$$

$$= \frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log\sqrt{ab}$$

ie., $\log_e k = \log\sqrt{ab}$

Thus $k = \sqrt{ab}$

32. Let $k = \lim_{x \to 0} [\sin^2(\pi/2 - x)]^{\sec^2(\pi/2 - x)} \ldots (1^\infty)$

Put $y = \pi/2 - x$ for convenience. As $x \to 0$, $y \to \pi/2$

Hence $k = \lim_{y \to \pi/2} (\sin^2 y)^{\sec^2 y} \ldots (1^\infty)$

$\Rightarrow$ $\log_e k = \lim_{y \to \pi/2} \sec^2 y \cdot \log(\sin^2 y) \ldots (\infty \times 0)$

$$= \lim_{y \to \pi/2} \frac{\log(\sin^2 y)}{\cos^2 y} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{y \to \pi/2} \frac{2\sin y \cos y/\sin^2 y}{-2\cos y \sin y} = -1$$

ie., $\log_e k = -1$

Thus $k = e^{-1} = 1/e$

33. Let $k = \lim_{x \to 1} (1 - x^2)^{1/\log(1-x)} \ldots (0^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 1} \dfrac{\log(1 - x^2)}{\log(1 - x)} \ldots \left(\dfrac{-\infty}{-\infty}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 1} \frac{-2x/1 - x^2}{-1/1 - x} = \lim_{x \to 1} \frac{2x(1 - x)}{(1 - x)(1 + x)}$$

ie., $\log_e k = \lim_{x \to 1} \dfrac{2x}{1 + x} = 1$

Thus $k = e^1 = e$

34. Let $k = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x \ldots (1^\infty)$

$$\Rightarrow \quad \log_e k = \lim_{x \to \infty} x \log\left(1 + \frac{a}{x}\right) \ldots (\infty \times 0)$$

$$= \lim_{x \to \infty} \frac{\log\left(1 + \frac{a}{x}\right)}{(1/x)} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to \infty} \frac{1\Big/\left(1 + \frac{a}{x}\right) \cdot -\frac{a}{x^2}}{(-1/x^2)} = a$$

ie., $\log_e k = a$

Thus $k = e^a$

35. Let $k = \lim_{x \to \infty} \left(\frac{ax + 1}{ax - 1}\right)^x$

We need to effect a basic simplification in this case of $x \to \infty$

ie., $$k = \lim_{x \to \infty} \left(\frac{a + (1/x)}{a - (1/x)}\right)^x \ldots (1^\infty)$$

Put $1/x = y$ for convenience, so that $y \to 0$ as $x \to \infty$

ie., $$k = \lim_{y \to 0} \left(\frac{a + y}{a - y}\right)^{1/y} \ldots (1^\infty)$$

$$\Rightarrow \quad \log_e k = \lim_{y \to 0} \frac{1}{y} \log\left(\frac{a + y}{a - y}\right)$$

$$= \lim_{y \to 0} \frac{\log(a + y) - \log(a - y)}{y} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{y \to 0} \frac{1/(a + y) + 1/(a - y)}{1} = \frac{1/a + 1/a}{1} = \frac{2}{a}$$

ie., $\log_e k = 2/a$

Thus $k = e^{2/a}$

36. Let $k = \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x^2} \quad \ldots (1^\infty)$

$$\Rightarrow \qquad \log_e k = \lim_{x \to 0} \frac{\log(\sin x/x)}{x^2} \ldots \left(\frac{0}{0} \right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{\frac{1}{(\sin x/x)} \cdot \frac{x \cos x - \sin x}{x^2}}{2x}$$

$$\log_e k = \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{x \cos x - \sin x}{2x^3}$$

$$= 1 \cdot \lim_{x \to 0} \frac{x \cos x - \sin x}{2x^3} \ldots \left(\frac{0}{0} \right)$$

$$= \lim_{x \to 0} \frac{-x \sin x + \cos x - \cos x}{6x^2}$$

ie., $$= \lim_{x \to 0} \frac{-1}{6} \cdot \frac{\sin x}{x} = \frac{-1}{6} \cdot 1 = \frac{-1}{6}$$

ie., $\log_e k = -1/6$

Thus $k = e^{-1/6}$

---

37. Let $k = \lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x} \quad \ldots (1^\infty)$

$$\Rightarrow \log_e k = \lim_{x \to 0} \frac{\log(\tan x/x)}{x} \ldots \left(\frac{0}{0} \right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{\frac{1}{(\tan x/x)} \cdot \frac{x \sec^2 x - \tan x}{x^2}}{1}$$

ie., $$= \lim_{x \to 0} \frac{x}{\tan x} \cdot \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{x^2}$$

$$= 1 \cdot \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{x^2} \ldots \left(\frac{0}{0} \right)$$

$$\log_e k = \lim_{x \to 0} \frac{x \cdot 2\sec^2 x \tan x + \sec^2 x - \sec^2 x}{2x}$$

$$= \lim_{x \to 0} \sec^2 x \cdot \tan x = 0$$

ie., $\log_e k = 0$

Thus $\mathbf{k = e^0 = 1}$

---

38. Let $k = \lim_{x \to 0} x^{\sin x} \ldots (0^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 0} \sin x \log x \ldots (0 \times -\infty)$

$$= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \ldots \left(\frac{-\infty}{\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x} = -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \tan x$$

ie., $\log_e k = -1 \cdot 0 = 0$

Thus $\mathbf{k = e^0 = 1}$

---

39. Let $k = \lim_{x \to 0} \left(\frac{1}{x}\right)^{2 \sin x} \ldots (\infty^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 0} 2\sin x \cdot \log(1/x) = -\lim_{x \to 0} 2\sin x \cdot \log x$

$= -2 \lim_{x \to 0} \sin x \log x = 0$ (*same as the previous example*)

$\therefore$ $\log_e k = 0$

Thus $\mathbf{k = e^0 = 1}$

---

40. Let $k = \lim_{x \to 0} (\cot x)^{\tan x} \ldots (\infty^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 0} \tan x \cdot \log(\cot x) \ldots (0 \times \infty)$

$$= \lim_{x \to 0} \frac{\log(\cot x)}{\cot x} \ldots \left(\frac{\infty}{\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{-\operatorname{cosec}^2 x/\cot x}{-\operatorname{cosec}^2 x} = \lim_{x \to 0} \tan x = 0$$

ie., $\log_e k = 0$

Thus $k = e^0 = 1$

---

41. Let $k = \lim_{x \to \infty} (\pi/2 - \tan^{-1} x)^{1/x} \ldots (0^0)$

ie., $k = \lim_{x \to \infty} (\cot^{-1} x)^{1/x} = \lim_{x \to \infty} [\tan^{-1}(1/x)]^{1/x}$

Put $1/x = y$. As $x \to \infty$, $y \to 0$

Hence $k = \lim_{y \to 0} (\tan^{-1} y)^y \ldots (0^{0^*})$

$\Rightarrow$ $\log_e k = \lim_{y \to 0} y \log(\tan^{-1} y) \ldots (0 \times -\infty)$

ie., $$= \lim_{y \to 0} \frac{\log(\tan^{-1} y)}{(1/y)} \ldots \left(\frac{-\infty}{\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{y \to 0} \frac{1/\tan^{-1} y \cdot 1/1 + y^2}{-1/y^2}$$

$$= \lim_{y \to 0} \frac{-y^2}{\tan^{-1} y \cdot (1 + y^2)} \ldots \left(\frac{0}{0}\right)$$

$$\log_e k = \lim_{y \to 0} \frac{-2y}{\tan^{-1} y \cdot 2y + 1} = \frac{0}{0+1} = 0$$

ie., $\log_e k = 0$

Thus $k = e^0 = 1$

Miscellaneous Examples

42. *Find the value of the constants a and b such that* $\lim\limits_{x\to 0} \dfrac{a\cosh x - b\cos x}{x^2}$ *may be equal to unity.*

>> Let $k = \lim\limits_{x\to 0} \dfrac{a\cosh x - b\cos x}{x^2} = \dfrac{a-b}{0}$

We must have $a - b = 0$ in order to apply the L' Hospital's rule,

Hence $k = \lim\limits_{x\to 0} \dfrac{a\sinh x + b\sin x}{2x} \ldots \left(\dfrac{0}{0}\right)$

$k = \lim\limits_{x\to 0} \dfrac{a\cosh x + b\cos x}{2} = \dfrac{a+b}{2}$

But we must have $k = 1$

$\therefore$ $(a+b)/2 = 1$ or $a+b=2$

By solving the equations, $a - b = 0$ and $a + b = 2$ we get $a = 1$, $b = 1$

Thus $a = 1, b = 1$

---

43. *Find the constants a, b, c such that* $\lim\limits_{x\to 0} \dfrac{a e^x - b\cos x + c e^{-x}}{x\sin x}$ *may be equal to 2.*

>> Let $k = \lim\limits_{x\to 0} \dfrac{a e^x - b\cos x + c e^{-x}}{x\sin x}$

ie., $= \lim\limits_{x\to 0} \dfrac{a e^x - b\cos x + c e^{-x}}{x^2} \cdot \lim\limits_{x\to 0} \dfrac{x}{\sin x}$

$= \lim\limits_{x\to 0} \dfrac{a e^x - b\cos x + c e^{-x}}{x^2} \cdot 1 = \dfrac{a-b+c}{0}$

Therefore we must have,

$$a - b + c = 0 \quad \ldots (1)$$

With (1) we have 0/0 form. Applying L' Hospital's rule,

$k = \lim\limits_{x\to 0} \dfrac{a e^x + b\sin x - c e^{-x}}{2x} = \dfrac{a-c}{0}$

Again we must have $a - c = 0$ to get 0/0 form. That is

$$a = c \quad \ldots (2)$$

Applying the rule again we have,

$$k = \lim_{x \to 0} \frac{a e^x + b \cos x + c e^{-x}}{2} = \frac{a+b+c}{2} \text{ and } k = 2 \text{ by data.}$$

ie., $(a+b+c)/2 = 2$ and therefore

$a+b+c=4 \qquad \ldots (3)$

Since $c = a$, (1) and (3) becomes $2a-b=0$ and $2a+b=4$

By solving we get $a = 1$ and $b = 2$. Hence $c = 1$

Thus $\boldsymbol{a = 1,\ b = 2,\ c = 1}$

---

44. *Find the value of the constant a such that* $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ *is finite. What is the finite limit ?*

>> Let $k = \lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3} \quad \ldots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{3x^2} = \frac{2+a}{0}$$

We must have $2+a=0$ or $a=-2$ for the $0/0$ form.

$$\therefore \quad k = \lim_{x \to 0} \frac{-4 \sin 2x - a \sin x}{6x} \quad \ldots \left(\frac{0}{0}\right)$$

$$k = \lim_{x \to 0} \frac{-8 \cos 2x - a \cos x}{6} = \frac{-(8+a)}{6} = -1 \text{ since } a = -2$$

Thus $\boldsymbol{a = -2}$ and **finite limit = -1**

---

45. *Evaluate* $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

>> Let $k = \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} \quad \ldots \left(\frac{0}{0}\right)$

We note that $\lim_{x \to 0} (1+x)^{1/x} = e$ and hence we have $0/0$ form.

Applying L'Hospital's rule,

$$k = \lim_{x \to 0} \frac{\frac{d}{dx}[(1+x)^{1/x}]}{1} = \lim_{x \to 0} \frac{du}{dx} \qquad \ldots (1)$$

where $u = (1+x)^{1/x}$ and let us find $\dfrac{du}{dx}$

Now $\log u = \dfrac{1}{x}\log(1+x) = \dfrac{\log(1+x)}{x}$

Differentiating w.r.t. x we get,

$$\frac{1}{u}\frac{du}{dx} = \frac{x \cdot \frac{1}{1+x} - \log(1+x)}{x^2}$$

$$\therefore \quad \frac{du}{dx} = u\left\{\frac{x-(1+x)\log(1+x)}{(1+x)x^2}\right\}$$

Now $\lim\limits_{x\to 0}\dfrac{du}{dx} = \lim\limits_{x\to 0} u \cdot \lim\limits_{x\to 0}\left\{\dfrac{x-(1+x)\log(1+x)}{x^2+x^3}\right\}$.

But $\lim\limits_{x\to 0} u = e$. By using (1) we have

$$k = e \cdot \lim_{x\to 0}\left\{\frac{x-(1+x)\log(1+x)}{x^2+x^3}\right\} \quad \cdots \left(\frac{0}{0}\right)$$

Hence $k = e\lim\limits_{x\to 0}\left\{\dfrac{1-1-\log(1+x)}{2x+3x^2}\right\} \quad \cdots \left(\dfrac{0}{0}\right)$

$$= e\lim_{x\to 0}\frac{-1/(1+x)}{2+6x} = e\cdot\frac{-1}{2} = \frac{-e}{2}$$

Thus $k = -e/2$

EXERCISES

Evaluate the following limits.

1. $\lim\limits_{x\to 0}\dfrac{x\cos x - \log(1+x)}{x^2}$

2. $\lim\limits_{x\to \pi/4}\dfrac{\sec^2 x - 2\tan x}{1+\cos 4x}$

3. $\lim\limits_{x\to 0}\dfrac{x-\log(1+x)}{1-\cos x}$

4. $\lim\limits_{x\to 0}\dfrac{\sin x\cdot\sin^{-1}x}{x^2}$

5. $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1+\sqrt{x-1}}{\sqrt{x^2-1}}$

6. $\lim\limits_{x\to 0}\dfrac{\log(1+kx^2)}{1-\cos x}$

7. $\lim_{x \to 1} \dfrac{\log(1-x)}{\cot \pi x}$

8. $\lim_{x \to \infty} \dfrac{x \cos(1/x)}{1+x}$

9. $\lim_{x \to 0} \log_x \tan x$

10. $\lim_{x \to \pi/2} \dfrac{\log(\cos x)}{\tan x}$

11. $\lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right]$

12. $\lim_{x \to \pi/2} (\sec x - \tan x)$

13. $\lim_{x \to 1} \left[\dfrac{1}{\log x} - \dfrac{x}{\log x} \right]$

14. $\lim_{x \to 0} x \log \tan x$

15. $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

16. $\lim_{x \to 1} (2-x)^{\tan(\pi x/2)}$

17. $\lim_{x \to 0} \left(\dfrac{a^x + b^x + c^x}{3} \right)^{1/x}$

18. $\lim_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x^2}$

19. Find the constants a and b such that $\lim_{x \to 0} \dfrac{x(1 + a\cos x) - b \sin x}{x^3}$ may be equal to unity.

20. If $\lim_{x \to 0} \dfrac{x(1 - a\cos x) + b \sin x}{x^2} = \dfrac{1}{3}$, show that the constants a and b satisfy the identity $a + b = 0$.

ANSWERS

1. 1/2	2. 1/2	3. 1	4. 1
5. $1/\sqrt{2}$	6. $2k$	7. 0	8. 1
9. 1	10. 0	11. 1/2	12. 0
13. -1	14. 0	15. $1/e$	16. $e^{2/\pi}$
17. $(abc)^{1/3}$	18. $e^{1/3}$	19. $a = -5/2,\ b = -3/2$	

2.2 Polar Curves

2.21 Introduction

We are conversant in representing the position of a point $P(x, y)$ in the *cartesian system* and accordingly (x, y) are called cartesian coordinates.

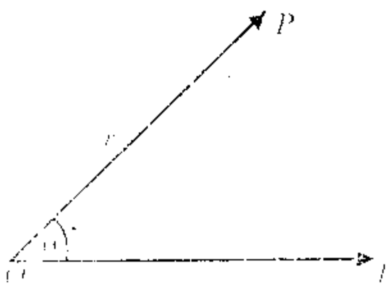
In this topic we discuss another important system to represent a point in a plane known as the *polar system.*

2.22 Polar Coordinates

Initial reference is chosen by spotting a point O in the plane called as the *pole.*

A line OL drawn through O is called the *initial line.* If P is any given point in the plane, join the points O and P with the result an angle is formed at O.

The length of OP denoted by r is called the *radius vector* of the point P and the angle $L O P$ denoted by θ measured in the anticlockwise direction is called the *vectorial angle.*



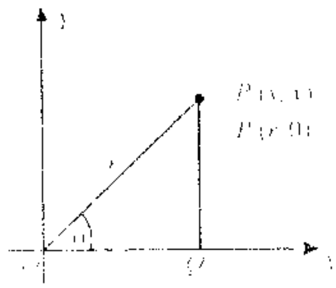
The pair r and θ represented by $P = (r, \theta)$ or $P(r, \theta)$ are called as the *polar coordinates* of the point P.
It is evident that r is positive and θ lies between 0 and 2π according as the position of the point P in the four quadrants.

We now proceed to establish the *relationship between the cartesian coordinates* (x, y) *and the polar coordinates* (r, θ).

Let (x, y) and (r, θ) respectively represent the cartesian and polar coordinates of any point P in the plane where the origin O is taken as the pole and the x-axis is taken as the initial line.

From the figure we have $OQ = x$, $PQ = y$.

Also from the right angled triangle OQP we have



$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} \quad \therefore \quad x = r\cos\theta \qquad \ldots (1)$$

$$\sin\theta = \frac{QP}{OP} = \frac{y}{r} \quad \therefore \quad y = r\sin\theta \qquad \ldots (2)$$

Further squaring and adding (1) and (2) we get

$$x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2 \cdot 1 = r^2$$

$$\therefore \quad r = \sqrt{x^2 + y^2} \qquad \ldots (3)$$

Also dividing (2) by (1) we get

$$\frac{r \sin \theta}{r \cos \theta} = \frac{y}{x} \quad ie., \quad \tan \theta = \frac{y}{x}$$

$$\therefore \quad \theta = \tan^{-1} \left(\frac{y}{x} \right) \qquad \ldots (4)$$

The relations (1) and (2) determine the cartesian coordinates in terms of polar coordinates whereas relations (3) and (4) determine the polar coordinates in terms of cartesian coordinates.

It is evident that r is a function of θ (r *depends on* θ) and the equation in the form

$$r = f(\theta) \quad \text{or} \quad f(r, \theta) = c, \ c \text{ being a constant}$$

is called the equation of the curve in the polar form or simply a *polar curve.*

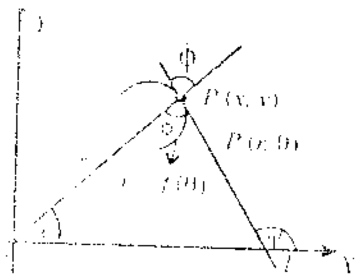
We now proceed to establish some results related to polar curves.

2.23 Angle between radius vector and tangent

Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

$$\therefore \quad X\hat{O}P = \theta \text{ and } OP = r.$$

Let PL be the tangent to the curve at P subtending an angle ψ with the positive direction of the initial line ($x - axis$) and ϕ be the angle between the radius vector OP and the tangent PL. That is $O\hat{P}L = \phi$.



From the figure we have

$$\psi = \phi + \theta$$

(*Recall from geometry that an exterior angle is equal to the sum of the interior opposite angles*)

$$\Rightarrow \quad \tan \psi = \tan (\phi + \theta)$$

$$\text{or} \quad \tan \psi = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \tan \theta} \qquad \ldots (1)$$

Let (x, y) be the cartesian coordinates of P so that we have,

$$x = r \cos \theta, \ y = r \sin \theta$$

Since r is a function of θ, we can as well regard these as parametric equations in terms of θ.

We also know from the geometrical meaning of the derivative that

$$\tan\psi = \frac{dy}{dx} = \text{ slope of the tangent } PL$$

ie., $$\tan\psi = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} \text{ since } x \text{ and } y \text{ are functions of } \theta.$$

ie., $$\tan\psi = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta} \text{ where } r' = \frac{dr}{d\theta}$$

(We try to correlate this expression with the already existing expression for $\tan\psi$ *in* (1). *Observe that the positive term in the denominator of* (1) *is equal to* 1)

Dividing both the numerator and denominator by $r'\cos\theta$ we have,

$$\tan\psi = \frac{\frac{r\cos\theta}{r'\cos\theta} + \frac{r'\sin\theta}{r'\cos\theta}}{\frac{-r\sin\theta}{r'\cos\theta} + \frac{r'\cos\theta}{r'\cos\theta}}$$

ie., $$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\cdot\tan\theta} \quad \ldots (2)$$

Comparing equations (1) and (2) we get

$$\tan\phi = \frac{r}{r'} = \frac{r}{\left(\frac{dr}{d\theta}\right)} \quad \text{or} \quad \boxed{\tan\phi = r\left(\frac{d\theta}{dr}\right)}$$

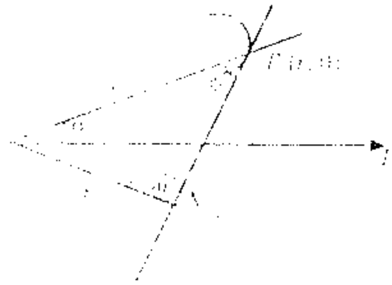
Equivalently we can write it in the form

$$\frac{1}{\tan\phi} = \frac{1}{r}\left(\frac{dr}{d\theta}\right) \quad \text{or} \quad \boxed{\cot\phi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)}$$

Note : *A question format :- Prove with usual notations* $\tan\phi = r\,\frac{d\theta}{dr}$

Length of the perpendicular from the pole to the tangent

Let O be the pole and OL be the initial line. Let $P(r, \theta)$ be any point on the curve and hence we have $OP = r$ and $L\hat{O}P = \theta$.



Draw $ON = p$ *(say)* perpendicular from the pole on the tangent at P and let ϕ be the angle made by the radius vector with the tangent.

From the figure $O\hat{N}P = 90^\circ$ and $L\hat{O}P = \theta$

Now from the right angled triangle ONP

$$\sin\phi = \frac{ON}{OP}$$

ie., $\sin\phi = \dfrac{p}{r}$ or $\boxed{p = r\sin\phi}$

(This expression is the basic expression for the length of the perpendicular p. We proceed to present the expression for p in terms of θ in two standard forms)

We have $p = r\sin\phi$ ... (1)

and $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$ ... (2)

Squaring equation (1) and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{1}{\sin^2\phi} \quad ie., \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

or $$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

Now using (2) we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

or $$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2} \quad \ldots (3)$$

Further, let $\dfrac{1}{r} = u$

Differentiating *w.r.t.* θ we get,

$$-\frac{1}{r^2}\left(\frac{dr}{d\theta}\right) = \frac{du}{d\theta} \Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2, \text{ by squaring.}$$

Thus (3) now becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \ldots (4)$$

Note : *The usual format of the question is as follows.*

(*i*) *Prove with usual notations* $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(*ii*) *Prove that for the curve* $r = f(\theta)$,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \text{ where } u = \frac{1}{r}$$

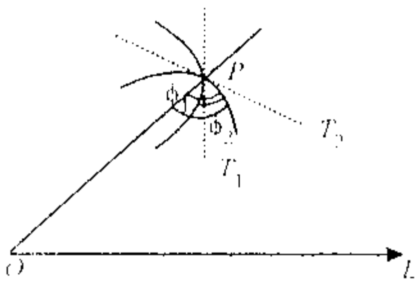
WORKED PROBLEMS

Angle of intersection of two polar curves

Basically we know that the angle of intersection of any two curves is equal to the angle between the tangents drawn at the point of intersection of the two curves.

Let $r = f_1(\theta)$ and $r = f_2(\theta)$ be two curves intersecting at the point P.

Let PT_1 and PT_2 be the tangents drawn to the curves at the point P.



It can be seen from the figure that ϕ_1 is the angle between the radius vector OP and the tangent PT_1 and ϕ_2 is the angle made by the radius vector OP with PT_2. It can be clearly seen that the angle between the two tangents is equal to $\phi_2 - \phi_1$

$\therefore$ *the acute angle of the intersection of the curves is equal to* $|\phi_2 - \phi_1|$

If $|\phi_2 - \phi_1| = \pi/2$ then we say that the two curves intersect *orthogonally*.

Further if $\phi_2 - \phi_1 = \frac{\pi}{2}$ then $\phi_2 = \frac{\pi}{2} + \phi_1$

$$\therefore \quad \tan\phi_2 = \tan\left(\frac{\pi}{2} + \phi_1\right) = -\cot\phi_1 = -\frac{1}{\tan\phi_1}$$

or $\quad$ $\tan\phi_1 \cdot \tan\phi_2 = -1.$

This result serves as an alternative condition for the orthogonality of two polar curves.

Working procedure for problems

- Given the equation in the form $r = f(\theta)$ we prefer to take logarithms first on both sides of the equation and then differentiate *w.r.t* θ which always gives the term $\frac{1}{r}\frac{dr}{d\theta}$ being the derivative of $\log r$ *w.r.t* θ.
- We directly substitute $\cot\phi$ or $\cot\phi_1$, $\cot\phi_2$ as the case may be for the term $\frac{1}{r}\frac{dr}{d\theta}$
- We simplify R.H.S too and try to put it in terms of cotangent *ie.*, "cot" so that we obtain ϕ or ϕ_1 and ϕ_2 as the case may be.
- $|\phi_2 - \phi_1|$ or $|\phi_1 - \phi_2|$ will give the angle of intersection.
- If this contains θ then we have to find θ by solving the pair of equations to obtain the angle of intersection independent of θ.
- Suppose we are not able to obtain ϕ_1 and ϕ_2 explicitly then we have to write the expressions for $\tan\phi_1$, $\tan\phi_2$ and use the formula

$$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

- If $\tan(\phi_1 - \phi_2) = \alpha$ (*say*) then the angle of intersection is equal to $\tan^{-1}(\alpha)$
- Also if $\tan\phi_1 \cdot \tan\phi_2 = -1$ then,

$$\tan(\phi_1 - \phi_2) = \infty \quad \Rightarrow \quad \phi_1 - \phi_2 = \pi/2$$

Note : *The following allied and compound angles trigonometric formulae will have frequent reference in problems.*

1. $\sin(\pi/2 - \theta) = \cos\theta \qquad \cos(\pi/2 - \theta) = \sin\theta$

   $\tan(\pi/2 - \theta) = \cot\theta \qquad \cot(\pi/2 - \theta) = \tan\theta$

2. $\sin(\pi/2 + \theta) = \cos\theta \qquad \cos(\pi/2 + \theta) = -\sin\theta$

   $\tan(\pi/2 + \theta) = -\cot\theta \qquad \cot(\pi/2 + \theta) = -\tan\theta$

3. $\tan(\pi/4 + \theta) = \frac{1 + \tan\theta}{1 - \tan\theta}$; $\cot(\pi/4 + \theta) = \frac{1 - \tan\theta}{1 + \tan\theta}$

Also we use the results :

$$1 + \cos\theta = 2\cos^2(\theta/2), \qquad 1 - \cos\theta = 2\sin^2(\theta/2),$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2), \qquad \cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$$

46. $\boldsymbol{r = a\,(1-\cos\theta)}$

Taking logarithms on both sides, $\log r = \log a + \log(1-\cos\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$$

ie., $$\cot\phi = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\left(\theta/2\right)$$

Thus $\cot\phi = \cot(\theta/2) \Rightarrow \boldsymbol{\phi = \theta/2}$

47. $\boldsymbol{r^2\cos 2\theta = a^2}$

Taking logarithms on both sides we have

$$2\log r + \log(\cos 2\theta) = 2\log a$$

Differentiating *w.r.t* θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{(-2\sin 2\theta)}{\cos 2\theta} = 0$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$

or $\cot\phi = \cot(\pi/2 - 2\theta) \Rightarrow \boldsymbol{\phi = \pi/2 - 2\theta}$

48. $\boldsymbol{r^m = a^m(\cos m\theta + \sin m\theta)}$

Taking logarithms on both sides we have, $m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{(-m\sin m\theta + m\cos m\theta)}{(\cos m\theta + \sin m\theta)}$$

ie., $$\frac{m}{r}\frac{dr}{d\theta} = \frac{m(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

Thus $$\cot\phi = \frac{\cos m\theta(1-\tan m\theta)}{\cos m\theta(1+\tan m\theta)} = \frac{1-\tan m\theta}{1+\tan m\theta}$$

or $\cot\phi = \cot(\pi/4 + m\theta) \Rightarrow \boldsymbol{\phi = \pi/4 + m\theta}$

49. $\quad l/r = 1 + e\cos\theta$

Taking logarithms on both sides we have,

$$\log l - \log r = \log(1 + e\cos\theta)$$

Differentiating *w.r.t* θ we get,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1 + e\cos\theta}$$

ie., $\quad \cot\phi = \dfrac{e\sin\theta}{1 + e\cos\theta}$ $\quad$ (*This cannot be simplified*)

or $\quad \tan\phi = \dfrac{1 + e\cos\theta}{e\sin\theta} \Rightarrow \boldsymbol{\phi = \tan^{-1}\left(\dfrac{1 + e\cos\theta}{e\sin\theta}\right)}$

Find the angle between the radius vector and the tangent and also find the slope of the tangent as indicated for the following curves.

50. $r = a(1 + \cos\theta)$ $\quad$ at $\theta = \pi/3$

51. $r\cos^2(\theta/2) = a$ $\quad$ at $\theta = 2\pi/3$

52. $2a/r = 1 - \cos\theta$ $\quad$ at $\theta = 2\pi/3$

53. $r = a(1 + \sin\theta)$ $\quad$ at $\theta = \pi/2$

50. $\quad \boldsymbol{r = a(1 + \cos\theta)}$

$\Rightarrow \quad \log r = \log a + \log(1 + \cos\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1 + \cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

Thus $\cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \boldsymbol{\phi = \pi/2 + \theta/2}$

At $\theta = \pi/3$, $\phi = \pi/2 + \pi/6$ or $\boldsymbol{\phi = 2\pi/3 = 120^\circ}$

Also we have $\psi = \theta + \phi = \pi/3 + 2\pi/3 = \pi = 180^\circ$

$\therefore$ slope of the tangent= $\tan\psi = \tan 180^\circ = 0$

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51. $\quad \boldsymbol{r\cos^2(\theta/2) = a}$

$\Rightarrow \quad \log r + 2\log\cos(\theta/2) = \log a$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} + 2\cdot\frac{-1/2\cdot\sin(\theta/2)}{\cos(\theta/2)} = 0$$

$\therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \tan(\theta/2)$

ie., $\cot\phi = \cot(\pi/2 - \theta/2) \quad \Rightarrow \quad \phi = \pi/2 - \theta/2$

At $\theta = 2\pi/3,\ \phi = \pi/2 - \pi/3 = \pi/6 = 30^\circ$

Also $\psi = \theta + \phi = 2\pi/3 + \pi/6 = 5\pi/6 = 150^\circ$

$\therefore$ slope of the tangent $= \tan\psi = \tan 150^\circ = -1/\sqrt{3}$

$(\because \tan(150^\circ) = \tan(90^\circ + 60^\circ) = -\cot 60^\circ = -1/\sqrt{3})$

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52. $2a/r = 1 - \cos\theta$

$\Rightarrow \quad \log 2a - \log r = \log(1 - \cos\theta)$

Differentiating *w.r.t* θ we get,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$$

ie., $-\cot\phi = \cot(\theta/2)$ or $\cot(-\phi) = \cot\theta/2 \Rightarrow \phi = -\theta/2$

At $\theta = 2\pi/3,\ \phi = -\pi/3 = -60^\circ$

Also $\psi = \theta + \phi = 2\pi/3 - \pi/3 = \pi/3 = 60^\circ$

$\therefore$ slope of the tangent $= \tan\psi = \tan(60^\circ) = \sqrt{3}$

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53. $r = a(1 + \sin\theta)$

$\Rightarrow \quad \log r = \log a + \log(1 + \sin\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta} \quad ie.,\ \cot\phi = \frac{\cos\theta}{1+\sin\theta}$$

At $\theta = \pi/2,\ \cot\phi = \frac{0}{1+1} = 0 \ \therefore\ \cot\phi = 0 \Rightarrow \phi = \pi/2$

Also $\psi = \theta + \phi = \pi/2 + \pi/2 = \pi$

$\therefore$ slope of the tangent $= \tan\psi = \tan\pi = 0$

Note : *We can simplify R.H.S and explicitly obtain ϕ as shown below.*

$$\cot\phi = \frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2) + \sin^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}$$

$$\cot\phi = \frac{[\cos(\theta/2)-\sin(\theta/2)]\,[\cos(\theta/2)+\sin(\theta/2)]}{[\cos(\theta/2)+\sin(\theta/2)]^2}$$

$$= \frac{[\cos(\theta/2)-\sin(\theta/2)]}{[\cos(\theta/2)+\sin(\theta/2)]} = \frac{\cos(\theta/2)\,[1-\tan(\theta/2)]}{\cos(\theta/2)\,[1+\tan(\theta/2)]}$$

ie., $$\cot\phi = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$$

Thus $\cot\phi = \cot(\pi/4+\theta/2) \Rightarrow \phi = \pi/4+\theta/2$

If we put $\theta = \pi/2$ we obtain $\phi = \pi/4+\pi/4 = \pi/2$

Show that the following pairs of curves intersect each other orthogonally

54. $r = a(1+\cos\theta)$ *and* $r = b(1-\cos\theta)$

55. $r = a(1+\sin\theta)$ *and* $r = a(1-\sin\theta)$

56. $r^n = a^n\cos n\theta$ *and* $r^n = a^n\sin n\theta$

57. $r^2\sin 2\theta = a^2$ *and* $r^2\cos 2\theta = b^2$

58. $r = 4\sec^2(\theta/2)$ *and* $r = 9\operatorname{cosec}^2(\theta/2)$

59. $r = ae^{\theta}$ *and* $re^{\theta} = b$

54. $r = a(1+\cos\theta)$: $r = b(1-\cos\theta)$

$\Rightarrow \log r = \log a + \log(1+\cos\theta)$: $\log r = \log b + \log(1-\cos\theta)$

Differentiating these w.r.t θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$$

$$\cot\phi_1 = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} \quad : \quad \cot\phi_2 = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

ie., $\cot\phi_1 = -\tan(\theta/2) = \cot(\pi/2+\theta/2)$: $\cot\phi_2 = \cot(\theta/2)$

$\Rightarrow \quad \phi_1 = \pi/2+\theta/2$: $\phi_2 = \theta/2$

$\therefore$ angle of intersection= $|\phi_1-\phi_2| = |\pi/2+\theta/2-\theta/2| = \pi/2$

Hence the curves intersect orthogonally.

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55. $r = a(1+\sin\theta) \ : \ r = a(1-\sin\theta)$

$\Rightarrow \quad \log r = \log a + \log(1+\sin\theta) \ : \ \log r = \log a + \log(1-\sin\theta)$

Differentiating these *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cos\theta}{1-\sin\theta}$$

ie., $$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta} \quad : \quad \cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$$

(Refering to the note in Ex-53, it requires quite a number of steps to obtain ϕ_1 and ϕ_2 explicitly in order to find $|\phi_1-\phi_2|$. But it will suffice if we can show that $\tan\phi_1 \cdot \tan\phi_2 = -1$)

We have $\tan\phi_1 = \dfrac{1+\sin\theta}{\cos\theta}$ and $\tan\phi_2 = \dfrac{1-\sin\theta}{-\cos\theta}$

$$\therefore \quad \tan\phi_1 \cdot \tan\phi_2 = \frac{1-\sin^2\theta}{-\cos^2\theta} = \frac{\cos^2\theta}{-\cos^2\theta} = -1$$

Hence the curves intersect orthogonally.

56. $r^n = a^n\cos n\theta \quad : \ r^n = b^n\sin n\theta$

Taking logarithms we have,

$$n\log r = n\log a + \log(\cos n\theta) \ : \ n\log r = n\log b + \log(\sin n\theta)$$

Differentiating these *w.r.t* θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} \quad : \quad \frac{n}{r}\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

ie., $$\cot\phi_1 = \cot(\pi/2+n\theta) \quad : \quad \cot\phi_2 = \cot n\theta$$

$$\Rightarrow \quad \phi_1 = \pi/2+n\theta \quad : \quad \phi_2 = n\theta$$

$$\therefore \quad |\phi_1-\phi_2| = |\pi/2+n\theta-n\theta| = \pi/2$$

Hence the curves intersect orthogonally.

57. $r^2 \sin 2\theta = a^2 \quad : \quad r^2 \cos 2\theta = b^2$

Taking logarithms we have,

$$2\log r + \log(\sin 2\theta) = 2\log a \; : \; 2\log r + \log(\cos 2\theta) = 2\log b$$

Differentiating these *w.r.t* θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0 \quad : \quad \frac{2}{r}\frac{dr}{d\theta} - \frac{2\sin 2\theta}{\cos 2\theta} = 0$$

ie., $\frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$

ie., $\cot\phi_1 = -\cot 2\theta \quad : \quad \cot\phi_2 = \tan 2\theta$

ie., $\cot\phi_1 = \cot(-2\theta) \quad : \quad \cot\phi_2 = \cot(\pi/2 - 2\theta)$

$\Rightarrow \quad \phi_1 = -2\theta \quad : \quad \phi_2 = \pi/2 - 2\theta$

$\therefore \quad |\phi_1 - \phi_2| = |-2\theta - \pi/2 + 2\theta| = \pi/2$

Hence the curves intersect orthogonally.

58. $r = 4\sec^2(\theta/2) \quad : \quad r = 9\,\mathrm{cosec}^2(\theta/2)$

Taking logarithms we have,

$$\log r = \log 4 + 2\log\sec(\theta/2) \; : \; \log r = \log 9 + 2\log\mathrm{cosec}(\theta/2)$$

Differentiating these *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2}{\sec(\theta/2)} \cdot \sec(\theta/2)\tan(\theta/2) \cdot \frac{1}{2}$$

$$: \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-2\,\mathrm{cosec}(\theta/2)\cot(\theta/2)}{\mathrm{cosec}(\theta/2)} \cdot \frac{1}{2}$$

ie., $\frac{1}{r}\frac{dr}{d\theta} = \tan(\theta/2) \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = -\cot(\theta/2)$

ie., $\cot\phi_1 = \cot(\pi/2 - \theta/2) \quad : \quad \cot\phi_2 = \cot(-\theta/2)$

$\Rightarrow \quad \phi_1 = \pi/2 - \theta/2 \quad : \quad \phi_2 = -\theta/2$

$\therefore \quad |\phi_1 - \phi_2| = |\pi/2 - \theta/2 + \theta/2| = \pi/2$

Hence the curves intersect orthogonally.

59. $r = ae^{\theta}$: $re^{\theta} = b$

$\Rightarrow$ $\log r = \log a + \theta \log e$: $\log r + \theta \log e = \log b$

But $\log e = 1$. Differentiating these *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \quad : \quad \frac{1}{r}\frac{dr}{d\theta} + 1 = 0$$

ie., $\cot \phi_1 = 1$: $\cot \phi_2 = -1$

$\Rightarrow$ $\phi_1 = \pi/4$: $\phi_2 = -\pi/4$ or $3\pi/4$

$\therefore$ $|\phi_1 - \phi_2| = |\pi/4 + \pi/4| = \pi/2$

Hence the curves intersect orthogonally.

Find the angle of intersection of the following pairs of curves.

60. $r = \sin\theta + \cos\theta$: $r = 2\sin\theta$

$\Rightarrow$ $\log r = \log(\sin\theta + \cos\theta)$: $\log r = \log 2 + \log(\sin\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}$$

ie., $\cot\phi_1 = \dfrac{\cos\theta(1 - \tan\theta)}{\cos\theta(1 + \tan\theta)}$: $\cot\phi_2 = \cot\theta \Rightarrow \phi_2 = \theta$

ie., $\cot\phi_1 = \cot(\pi/4 + \theta) \Rightarrow \phi_1 = \pi/4 + \theta$

$\therefore$ $|\phi_1 - \phi_2| = |\pi/4 + \theta - \theta| = \pi/4$

The angle of intersection is $\pi/4$

61. $r = a\log\theta$: $r = a/\log\theta$

$\Rightarrow \quad \log r = \log a + \log(\log\theta)$: $\log r = \log a - \log(\log\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\log\theta \cdot \theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log\theta \cdot \theta}$$

ie., $$\cot\phi_1 = \frac{1}{\theta\log\theta} \quad : \quad \cot\phi_2 = -\frac{1}{\theta\log\theta}$$

Note : *we cannot find* ϕ_1 *and* ϕ_2 *explicitly.*

$\therefore \quad \tan\phi_1 = \theta\log\theta$: $\tan\phi_2 = -\theta\log\theta$

Now consider, $\tan(\phi_1 - \phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2}$

ie., $$\tan(\phi_1 - \phi_2) = \frac{2\theta\log\theta}{1 - (\theta\log\theta)^2} \qquad \ldots (1)$$

We have to find θ by solving the given pair of equations : $r = a\log\theta$ and $r = a/\log\theta$.

Equating the R.H.S we have $a\log\theta = \dfrac{a}{\log\theta}$

ie., $(\log\theta)^2 = 1$ or $\log\theta = 1 \Rightarrow \theta = e$

Substituting $\theta = e$ in (1) we get,

$$\tan(\phi_1 - \phi_2) = \frac{2e}{1 - e^2} \quad (\because \log e = 1)$$

$\therefore$ **angle of intersection** $= \phi_1 - \phi_2 = \tan^{-1}\left(\dfrac{2e}{1 - e^2}\right) = 2\tan^{-1} e$

62. $r^2\sin 2\theta = 4$: $r^2 = 16\sin 2\theta$

$2\log r + \log(\sin 2\theta) = \log 4$: $2\log r = \log 16 + \log(\sin 2\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0 \quad : \quad \frac{2}{r}\frac{dr}{d\theta} = \frac{2\cos 2\theta}{\sin 2\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \cot 2\theta$$

ie., $\cot\phi_1 = \cot(-2\theta)$: $\cot\phi_2 = \cot 2\theta$

$\Rightarrow$ $\phi_1 = -2\theta$: $\phi_2 = 2\theta$

$\therefore$ $|\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$ ... (1)

Now consider $r^2 = \dfrac{4}{\sin 2\theta}$ and $r^2 = 16\sin 2\theta$

$\therefore$ $\dfrac{4}{\sin 2\theta} = 16.\sin 2\theta$ or $4\sin^2 2\theta = 1$

ie., $\sin^2 2\theta = 1/4$ or $\sin 2\theta = 1/2 \Rightarrow 2\theta = \pi/6$ $\therefore$ $\theta = \pi/12$

Substituting $\theta = \pi/12$ in (1) we get $|\phi_1 - \phi_2| = \pi/3$

$\therefore$ **angle of intersection** $= \pi/3 = 60^\circ$

63. $r = a(1-\cos\theta)$: $r = 2a\cos\theta$

Taking logarithms we have,

$\log r = \log a + \log(1-\cos\theta)$: $\log r = \log 2a + \log(\cos\theta)$

Differentiating these *w.r.t* θ, we get,

$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1-\cos\theta}$: $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-\sin\theta}{\cos\theta}$

ie., $\cot\phi_1 = \dfrac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$: $\cot\phi_2 = -\tan\theta$

ie., $\cot\phi_1 = \cot(\theta/2)$: $\cot\phi_2 = \cot(\pi/2+\theta)$

$\Rightarrow$ $\phi_1 = \theta/2$: $\phi_2 = \pi/2+\theta$

$\therefore$ $|\phi_1 - \phi_2| = |\theta/2 - \pi/2 - \theta| = \pi/2 + \theta/2$ ... (1)

Now consider $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$\therefore$ $a(1-\cos\theta) = 2a\cos\theta$

or $3\cos\theta = 1$ or $\theta = \cos^{-1}(1/3)$

Substituting this value in (1) we get,

the angle of intersection $= \pi/2 + 1/2 \cdot \cos^{-1}(1/3)$

64. $r = 6\cos\theta$: $r = 2(1+\cos\theta)$

$\Rightarrow$ $\log r = \log 6 + \log(\cos\theta)$: $\log r = \log 2 + \log(1+\cos\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$$

ie., $\cot\phi_1 = -\tan\theta$: $\cot\phi_2 = \dfrac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$

ie., $\cot\phi_1 = \cot(\pi/2+\theta)$: $\cot\phi_2 = -\tan(\theta/2) = \cot(\pi/2+\theta/2)$

$\Rightarrow$ $\phi_1 = \pi/2+\theta$: $\phi_2 = \pi/2+\theta/2$

$\therefore$ $|\phi_1 - \phi_2| = \theta/2$ ... (1)

Equating the R.H.S of the given equations we have

$$6\cos\theta = 2(1+\cos\theta) \text{ or } \cos\theta = 1/2 \Rightarrow \theta = \pi/3$$

$\therefore$ from (1) $|\phi_1 - \phi_2| = \pi/6 = 30^\circ$

Hence the angle of intersection $= \pi/6 = 30^\circ$

65. $r^n = a^n \sec(n\theta+\alpha)$: $r^n = b^n \sec(n\theta+\beta)$

Taking logarithms we have,

$$n\log r = n\log a + \log\sec(n\theta+\alpha) : n\log r = n\log b + \log\sec(n\theta+\beta)$$

Differentiating these *w.r.t* θ, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n\sec(n\theta+\alpha)\tan(n\theta+\alpha)}{\sec(n\theta+\alpha)} \quad : \quad \frac{n}{r}\frac{dr}{d\theta} = \frac{n\sec(n\theta+\beta)\tan(n\theta+\beta)}{\sec(n\theta+\beta)}$$

ie., $\dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(n\theta+\alpha)$: $\dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(n\theta+\beta)$

ie., $\cot\phi_1 = \cot[\pi/2 - (n\theta+\alpha)]$: $\cot\phi_2 = \cot[\pi/2-(n\theta+\beta)]$

$\Rightarrow$ $\phi_1 = \pi/2 - n\theta - \alpha$: $\phi_2 = \pi/2 - n\theta - \beta$

$\therefore$ $|\phi_1 - \phi_2| = |-\alpha+\beta| = \alpha-\beta$, where $\alpha > \beta$

Hence the angle of intersection $= \alpha - \beta$, where $\alpha > \beta$

66. $r = a(1+\cos\theta)$: $r^2 = a^2\cos 2\theta$

Taking logarithms we have,

$$\log r = \log a + \log(1+\cos\theta) \quad : \quad 2\log r = 2\log a + \log(\cos 2\theta)$$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} \quad : \quad \frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}$$

ie., $\cot\phi_1 = \dfrac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$: $\cot\phi_2 = -\tan 2\theta$

ie., $\cot\phi_1 = -\tan(\theta/2)$: $\cot\phi_2 = \cot(\pi/2+2\theta)$

ie., $\cot\phi_1 = \cot(\pi/2+\theta/2)$: $\Rightarrow \phi_2 = \pi/2+2\theta$

$\Rightarrow$ $\phi_1 = \pi/2+\theta/2$

$\therefore$ $|\phi_1-\phi_2| = |\pi/2+\theta/2-\pi/2-2\theta| = 3\theta/2$ ... (1)

Now, squaring the first of the given equations and then equating the R.H.S of the two equations we have

$$a^2(1+\cos\theta)^2 = a^2\cos 2\theta$$

ie., $1+2\cos\theta+\cos^2\theta = 2\cos^2\theta-1$

or $\cos^2\theta-2\cos\theta-2 = 0$

$\therefore$ $\cos\theta = \dfrac{2\pm\sqrt{4+8}}{2} = \dfrac{2\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$

Since $\cos\theta$ cannot exceed 1 numerically we have to take

$$\cos\theta = 1-\sqrt{3} \Rightarrow \theta = \cos^{-1}(1-\sqrt{3})$$

Otherwise we can also have,

$$1-2\sin^2(\theta/2) = 1-\sqrt{3} \text{ or } \sin^2(\theta/2) = \sqrt{3}/2 = \sqrt{3/4}$$

ie., $\sin\left(\theta/2\right) = (\sqrt{3/4})^{1/2} = (3/4)^{1/4}$

$\therefore$ $\theta/2 = \sin^{-1}\left(3/4\right)^{1/4}$

Substituting this value in (1) we get,

the angle of intersection $= 3\sin^{-1}(3/4)^{1/4}$

67. $r = a\theta/1+\theta \quad : \quad r = a/1+\theta^2$

Taking logarithms we have,

$$\log r = \log a + \log\theta - \log(1+\theta) \quad : \quad \log r = \log a - \log(1+\theta^2)$$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} - \frac{1}{1+\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2}$$

ie., $\cot\phi_1 = \dfrac{1}{\theta(1+\theta)} \quad : \quad \cot\phi_2 = \dfrac{-2\theta}{1+\theta^2}$

$\Rightarrow \quad \tan\phi_1 = \theta+\theta^2 \quad : \quad \tan\phi_2 = \dfrac{1+\theta^2}{-2\theta}$

Also by equating the R.H.S of the given equations we have,

$$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$$

or $\quad \theta+\theta^3 = 1+\theta \quad$ or $\quad \theta^3 = 1 \Rightarrow \theta = 1$

$\therefore \quad \tan\phi_1 = 2$ and $\tan\phi_2 = -1$ at $\theta = 1$

Consider, $\tan(\phi_1 - \phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$

$\therefore \quad \tan(\phi_1-\phi_2) = \dfrac{2-(-1)}{1+(-2)} = -3$

Taking the absolute value, **the angle of intersection** $= \tan^{-1}(3)$

---

68. $r = a\theta \quad : \quad r = a/\theta$

$\Rightarrow \quad \log r = \log a + \log\theta \quad : \quad \log r = \log a - \log\theta$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta}$$

ie., $\cot\phi_L = \dfrac{1}{\theta} \quad : \quad \cot\phi_2 = -\dfrac{1}{\theta}$

or $\quad \tan\phi_1 = \theta \quad : \quad \tan\phi_2 = -\theta$

Also by equating the R.H.S of the given equations we have

$$a\theta = a/\theta \text{ or } \theta^2 = 1 \Rightarrow \theta = \pm 1$$

When $\theta = 1,\ \tan\phi_1 = 1,\ \tan\phi_2 = -1$ and

when $\theta = -1,\ \tan\phi_1 = -1,\ \tan\phi_2 = 1.$

$\therefore \quad \tan\phi_1 \cdot \tan\phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \pi/2$

The curves intersect at right angles.

>> We have $r^2 = a^2 \cos 2\theta$

$\Rightarrow \quad 2\log r = 2\log a + \log(\cos 2\theta)$

Differentiating *w.r.t* θ, we have,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$$

ie., $\quad \cot\phi = \cot(\pi/2 + 2\theta) \Rightarrow \phi = \pi/2 + 2\theta$

If ψ is the angle made by the tangent with the initial line, $\psi - (\pi/2)$ will be the angle made by the normal with the initial line.

We know that $\psi = \phi + \theta = (\pi/2 + 2\theta) + \theta = \pi/2 + 3\theta$

Hence $\psi = \pi/2 + 3\theta \quad \Rightarrow \quad \psi - (\pi/2) = 30$

Thus $(\pi/2) + 3\theta$ and 3θ are respectively the angles made by the tangent and the normal with the initial line.

>> We have $r = a(1 + \cos\theta)$

$\Rightarrow \quad \log r = \log a + \log(1 + \cos\theta)$

Differentiating *w.r.t* θ, we have,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

ie., $\quad \cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \phi = \pi/2 + \theta/2$.

If ψ is the angle made by the tangent with the initial line then,

$$\psi = \phi + \theta = \pi/2 + 3\theta/2$$

At $\quad \theta = \pi/3 : \psi = \pi/2 + \pi/2 = \pi = 180^0$

At $\theta = 2\pi/3 \; : \; \psi = \pi/2 + \pi = 3\pi/2 = 270^\circ$

Hence we conclude that the tangents to the given curve at $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively **parallel and perpendicular to the initial line.**

>> We have with usual notations $\psi = \theta + \phi$

$\therefore \quad \phi = \psi - \theta \Rightarrow \tan\phi = \tan(\psi - \theta)$

or $\quad \tan\phi = \dfrac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta}$ . . . (1)

We also have $\tan\psi = \dfrac{dy}{dx} = y'$ and

$x = r\cos\theta,\; y = r\sin\theta$ gives $\tan\theta = (y/x)$

Substituting these in (1) we get,

$$\tan\phi = \frac{y' - (y/x)}{1 + y'(y/x)} = \frac{xy' - y}{x + yy'}$$

Thus $\quad \boldsymbol{\tan\phi = \dfrac{xy' - y}{x + yy'}}$

In the context of deriving an expression for the length of the perpendicular (p) from the pole to the tangent we obtained the expression in the form $p = r\sin\phi$.

The equation of the given curve $r = f(\theta)$ expressed in terms of p and r is called as the pedal equation or $p - r$ equation of the curve $r = f(\theta)$.

Remark : *Many equations of the standard cartesian curves $y = f(x)$ are expressible in the parametric form $x = f_1(t)$, $y = f_2(t)$. Eliminating t we get $y = f(x)$. We have a similar concept in respect of $r = f(\theta)$.*

Working procedure for finding the pedal equation of a polar curve

- Given $r = f(\theta)$ we first obtain ϕ.
- We substitute ϕ (*usually a function of* θ) into the equation $p = r\sin\phi$ so that this equation assumes the form $p = r\,g(\theta)$
- We need to eliminate θ between the equations :

$$r = f(\theta) \quad \ldots (1)$$

$$p = r\,g(\theta) \quad \ldots (2)$$

This will give us an equation in p and r being the required pedal equation.

➲ It may be noted that if we are unable to obtain ϕ explicitly in terms of θ, we have to sqaure and take the reciprocal of $p = r\sin\phi$.

This will give us :

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}[1 + \cot^2\phi]$$

We substitute for $\cot\phi$ itself in terms of θ. Elimination of θ by using the given equation will give us the pedal equation.

Find the pedal equation of the following curves :

72. $2a/r = (1 + \cos\theta)$
73. $r(1 - \cos\theta) = 2a$
74. $r^2 = a^2\sec 2\theta$
75. $r^n = a^n\cos n\theta$
76. $r^m = a^m(\cos m\theta + \sin m\theta)$
77. $r = 2(1 + \cos\theta)$
78. $l/r = 1 + e\cos\theta$
79. $r^n = a^n\operatorname{sech} n\theta$

72. $\dfrac{2a}{r} = 1 + \cos\theta$

$\Rightarrow \quad \log 2a - \log r = \log(1 + \cos\theta)$

Differentiating *w.r.t* θ, we get,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

ie., $\quad \cot\phi = \cot(\pi/2 - \theta/2) \Rightarrow \phi = \pi/2 - \theta/2$

Consider $p = r\sin\phi$ and substituting the value of ϕ we have

$$p = r\sin(\pi/2 - \theta/2) = r\cos(\theta/2)$$

Now we have $\dfrac{2a}{r} = 1 + \cos\theta$ ... (1)

$$p = r\cos(\theta/2) \quad \text{... (2)}$$

We have to eliminate θ from (1) and (2)

(It will be convenient for elimination if we can have similar functions of θ in the R.H.S of the two equations)

(1) can be put in the form $\dfrac{2a}{r} = 2\cos^2(\theta/2)$ or $\dfrac{a}{r} = \cos^2(\theta/2)$

Also from (2), $\dfrac{p}{r} = \cos(\theta/2)$

Hence we get, $\frac{a}{r} = \left(\frac{p}{r}\right)^2$ or $\frac{a}{r} = \frac{p^2}{r^2}$ or $p^2 = ar$

Thus $p^2 = ar$ is the required pedal equation.

---

73. $r(1-\cos\theta) = 2a$

$\Rightarrow \quad \log r + \log(1-\cos\theta) = \log 2a$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \text{ or } \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\therefore \quad \cot\phi = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -\cot(\theta/2)$$

ie., $\quad \cot\phi = \cot(-\theta/2) \Rightarrow \phi = -(\theta/2)$

Consider $p = r\sin\phi$

$\therefore \quad p = r\sin(-\theta/2)$ or $p = -r\sin(\theta/2)$

Now we have, $\quad r(1-\cos\theta) = 2a \quad \ldots(1)$

$\quad p = -r\sin(\theta/2) \quad \ldots(2)$

We have to eliminate θ from (1) and (2).

(1) can be put in the form $r\cdot 2\sin^2(\theta/2) = 2a$

ie., $\quad r\sin^2(\theta/2) = a.$

But $p/-r = \sin(\theta/2)$, from (2).

$$\therefore \quad r\left(\frac{p^2}{r^2}\right) = a \text{ or } p^2 = ar$$

Thus $p^2 = ar$ is the required pedal equation.

---

74. $r^2 = a^2\sec 2\theta$

$\Rightarrow \quad 2\log r = 2\log a + \log(\sec 2\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{2\sec 2\theta\tan 2\theta}{\sec 2\theta} \quad ie., \quad \frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$

ie., $\quad \cot\phi = \cot(\pi/2 - 2\theta) \Rightarrow \phi = \pi/2 - 2\theta$

Consider $p = r\sin\phi$ $\therefore$ $p = r\sin(\pi/2 - 2\theta)$ *ie.,* $p = r\cos 2\theta$

Now we have, $r^2 = a^2 \sec 2\theta$ ... (1)

$p = r \cos 2\theta$ ... (2)

From (2), $p/r = \cos 2\theta$ or $r/p = \sec 2\theta$

Substituting in (1) we get, $r^2 = a^2 (r/p)$ or $pr = a^2$

Thus $\boldsymbol{pr = a^2}$ is the required pedal equation.

75. $\boldsymbol{r^n = a^n \cos n\theta}$

$\Rightarrow \quad n \log r = n \log a + \log(\cos n\theta)$.

Differentiating *w.r.t* θ, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} \quad ie., \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$\therefore \quad \cot\phi = \cot(\pi/2 + n\theta) \Rightarrow \phi = \pi/2 + n\theta$

Consider $p = r \sin\phi$

$\therefore \quad p = r \sin(\pi/2 + n\theta)$ *ie.,* $p = r \cos n\theta$

Now we have, $r^n = a^n \cos n\theta$ ... (1)

$p = r \cos n\theta$ ... (2)

$\therefore$ (1) as a consequence of (2) is $r^n = a^n (p/r)$

Thus $\boldsymbol{r^{n+1} = p\,a^n}$ is the required pedal equation.

76. $\boldsymbol{r^m = a^m(\cos m\theta + \sin m\theta)}$

$\Rightarrow \quad m \log r = m \log a + \log(\cos m\theta + \sin m\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{-m \sin m\theta + m \cos m\theta}{\cos m\theta + \sin m\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \frac{\cos m\theta(1 - \tan m\theta)}{\cos m\theta(1 + \tan m\theta)}$$

$\therefore \quad \cot\phi = \cot(\pi/4 + m\theta) \Rightarrow \phi = \pi/4 + m\theta$

Consider $p = r \sin\phi$

$\therefore \quad p = r \sin(\pi/4 + m\theta)$

ie., $\quad p = r[\sin(\pi/4)\cos m\theta + \cos(\pi/4)\sin m\theta]$

ie., $p = \frac{r}{\sqrt{2}} (\cos m\theta + \sin m\theta)$

(We have used the formula of $\sin (A + B)$ *and also the values* $\sin (\pi/4) = \cos (\pi/4) = 1/\sqrt{2}$)

Now we have, $r^m = a^m (\cos m \theta + \sin m \theta)$ . . . (1)

$$p = \frac{r}{\sqrt{2}} (\cos m \theta + \sin m \theta) \quad \ldots (2)$$

Using (2) in (1) we get,

$$r^m = a^m \cdot \frac{p\sqrt{2}}{r} \quad \text{or} \quad r^{m+1} = \sqrt{2}\, a^m p$$

Thus $r^{m+1} = \sqrt{2}\, a^m p$ **is the required pedal equation.**

77. $r = 2 (1 + \cos\theta)$

$\Rightarrow$ $\log r = \log 2 + \log (1 + \cos\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

$\therefore$ $\cot\phi = \cot (\pi/2 + \theta/2) \Rightarrow \phi = \pi/2 + \theta/2$

Consider $p = r\sin\phi$

$\therefore$ $p = r\sin(\pi/2 + \theta/2) = r\cos(\theta/2)$

Now we have, $r = 2 (1 + \cos\theta)$ . . . (1)

$$p = r\cos(\theta/2) \quad \ldots (2)$$

(1) can be put in the form $r = 2 \cdot 2\cos^2(\theta/2)$

ie., $r = 4\cos^2(\theta/2)$

From (2), $p/r = \cos(\theta/2)$ and hence (1) becomes,

$$r = 4 \cdot (p^2/r^2) \quad \text{or} \quad r^3 = 4p^2$$

Thus $r^3 = 4p^2$ is the required pedal equation.

78. $l/r = 1 + e\cos\theta$

$\Rightarrow$ $\log l - \log r = \log (1 + e\cos\theta)$

Differentiating *w.r.t* θ, we get,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$$

ie., $\cot\phi = \dfrac{e\sin\theta}{1+e\cos\theta}$ We cannot find ϕ explicitly.

Consider $p = r\sin\phi$

By squaring and taking the reciprocal we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

Substituting for $\cot\phi$ itself we have

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right\} \quad \ldots (1)$$

Also we have $\dfrac{l}{r} = 1+e\cos\theta$ ... (2)

We need to eliminate θ from (1) and (2).

From (2) $\dfrac{l}{r}-1 = e\cos\theta$ ... (3)

Also $e^2\sin^2\theta = e^2(1-\cos^2\theta) = e^2-e^2\cos^2\theta$

By using (3) we have $e^2\sin^2\theta = e^2-\left(\dfrac{l}{r}-1\right)^2$ ... (4)

Now substituting (3) and (4) in (1) we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2-\left(\frac{l}{r}-1\right)^2}{(l^2/r^2)}\right\}$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left\{e^2-\left(\frac{l}{r}-1\right)^2\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left\{e^2-\frac{l^2}{r^2}+\frac{2l}{r}-1\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{e^2}{l^2}-\frac{1}{r^2}+\frac{2}{lr}-\frac{1}{l^2}$$

Thus $\boldsymbol{\dfrac{1}{p^2} = \dfrac{e^2-1}{l^2}+\dfrac{2}{lr}}$ is the required pedal equation.

79. $r^n = a^n \operatorname{sech} n\theta$

$\Rightarrow \quad n \log r = n \log a + \log(\operatorname{sech} n\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \operatorname{sech} n\theta \tanh n\theta}{\operatorname{sech} n\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -n \tanh n\theta$$

$\therefore \quad \cot\phi = -\tanh n\theta$ and ϕ cannot be found explicitly.

Consider $p = r\sin\phi$. Squaring and taking the reciprocal, we get

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\tan h^2 n\theta) \qquad \ldots (1)$$

Also we have, $r^n = a^n \operatorname{sech} n\theta$

$\therefore \quad \dfrac{r^n}{a^n} = \operatorname{sech} n\theta$ and we have $1 - \tanh^2 n\theta = \operatorname{sech}^2 n\theta$

$$\therefore \quad \tanh^2 n\theta = 1 - \operatorname{sech}^2 n\theta = 1 - \left(\frac{r^n}{a^n}\right)^2$$

Substituting this expression in the R.H.S of (1) we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{2 - \frac{r^{2n}}{a^{2n}}\right\}$$ being the required pedal equation.

---

80. *For the equiangular spiral $r = a e^{\theta \cot \alpha}$, a and α are constants show that the tangent is inclined at a constant angle with the radius vector and hence find the pedal equation of the curve*

\>> We have $r = a\, e^{\theta \cot\alpha}$

$\Rightarrow \quad \log r = \log a + \theta \cot\alpha \log e \quad$ But $\log e = 1$

$\therefore \quad \log r = \log a + \cot\alpha \cdot \theta$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha \cdot 1$$

ie., $\cot\phi = \cot\alpha \Rightarrow \phi = \alpha = \text{constant}$

$\therefore$ **the tangent is inclined at a constant angle with the radius vector.**

Consider $p = r\sin\phi$. But $\phi = \alpha$

$\therefore \quad p = r\sin\alpha$. This is independent of θ.

Hence $p = r\sin\alpha$ is the required pedal equation.

---

81. *Show that for the curve* $r\cos(\sqrt{a^2-b^2}/a)\theta = \sqrt{a^2-b^2}$, $p^2(r^2+b^2) = a^2r^2$

>> We have $r\cos(\sqrt{a^2-b^2}/a)\theta = \sqrt{a^2-b^2}$

For convenience let $\sqrt{a^2-b^2}/a = k$, a constant.

We now have $r\cos k\theta = ka$

$\Rightarrow \quad \log r + \log(\cos k\theta) = \log(ka)$.

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{-k\sin k\theta}{\cos k\theta} = 0$$

ie., $\cot\phi = k\tan k\theta$. We cannot find ϕ explicitly.

Consider $p = r\sin\phi$.

Squaring and taking the reciprocal, we have

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \qquad \text{or} \qquad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}(1+k^2\tan^2 k\theta) \qquad \ldots(1)$$

But $r\cos k\theta = ka$ $\qquad \ldots(2)$

We need to eliminate θ from (1) and (2).

From (2), $\cos k\theta = \dfrac{ka}{r} \Rightarrow \sec k\theta = \dfrac{r}{ka}$

Now $\tan^2 k\theta = \sec^2 k\theta - 1 = \dfrac{r^2}{k^2a^2} - 1$

Substituting this expression in (1) we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + k^2\left(\frac{r^2}{k^2a^2} - 1\right)\right\}$$

ie., $\quad \dfrac{1}{p^2} = \dfrac{1}{r^2}\left\{1 + \dfrac{r^2}{a^2} - k^2\right\}$ But $k^2 = \dfrac{a^2 - b^2}{a^2}$

$\therefore \quad \dfrac{1}{p^2} = \dfrac{1}{r^2}\left\{1 + \dfrac{r^2}{a^2} - \dfrac{a^2 - b^2}{a^2}\right\}$

ie., $\quad \dfrac{1}{p^2} = \dfrac{1}{r^2}\left\{\dfrac{a^2 + r^2 - a^2 + b^2}{a^2}\right\}$

ie., $\quad \dfrac{1}{p^2} = \dfrac{r^2 + b^2}{r^2 a^2}$

Thus $\quad p^2 (r^2 + b^2) = a^2 r^2$ as required.

82. *Find the value of* ϕ *for the curve* $a\theta = \sqrt{r^2 - a^2} - a\cos^{-1}(a/r)$

Note : *Observing the complexity of the given equation we do not venture to take logarithms.*

>> We have $a\theta = \sqrt{r^2 - a^2} - a\cos^{-1}(a/r)$

Differentiating *w.r.t* θ on both sides keeping in mind that r is a function of θ we obtain

$$a = \frac{1}{2\sqrt{r^2 - a^2}} \cdot 2r\frac{dr}{d\theta} - a \cdot -\frac{1}{\sqrt{1 - (a^2/r^2)}} \cdot -\frac{a}{r^2}\frac{dr}{d\theta}$$

ie., $\quad a = \dfrac{r}{\sqrt{r^2 - a^2}} \cdot \dfrac{dr}{d\theta} - \dfrac{a^2}{r^2}\,\dfrac{r}{\sqrt{r^2 - a^2}}\,\dfrac{dr}{d\theta}$

ie., $\quad a = \dfrac{r}{\sqrt{r^2 - a^2}} \cdot \dfrac{dr}{d\theta}\left(1 - \dfrac{a^2}{r^2}\right)$

ie., $\quad a = \dfrac{r}{\sqrt{r^2 - a^2}} \cdot \dfrac{dr}{d\theta}\left(\dfrac{r^2 - a^2}{r^2}\right)$

ie., $\quad a = \dfrac{\sqrt{r^2 - a^2}}{r}\,\dfrac{dr}{d\theta}$ or $\quad r\dfrac{d\theta}{dr} = \dfrac{\sqrt{r^2 - a^2}}{a}$

ie., $\quad \tan\phi = \dfrac{\sqrt{r^2 - a^2}}{a} \quad \therefore \quad \boldsymbol{\phi = \tan^{-1}\left(\dfrac{\sqrt{r^2 - a^2}}{a}\right)}$

83. *Establish the pedal equation of the curve* $r^n = a^n \sin n\theta + b^n \cos n\theta$ *in the form* $p^2 (a^{2n} + b^{2n}) = r^{2n+2}$

>> We have $r^n = a^n \sin n\theta + b^n \cos n\theta$

$\Rightarrow \quad n\log r = \log(a^n \sin n\theta + b^n \cos n\theta)$

Differentiating *w.r.t* θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{na^n \cos n\theta - nb^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Dividing by n, $$\cot\phi = \frac{a^n \cos n\theta - b^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Consider $p = r\sin\phi$

Since ϕ cannot be found, squaring and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{(a^n\cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{(a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^{2n}(\sin^2 n\theta + \cos^2 n\theta) + b^{2n}(\cos^2 n\theta + \sin^2 n\theta)}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

(Product terms cancels out in the numerator)

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{a^{2n}+b^{2n}}{(a^n \sin n\theta + b^n \cos n\theta)^2}$$

or $$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{a^{2n}+b^{2n}}{(r^n)^2}\text{, by using the given equation.}$$

Thus $p^2(a^{2n}+b^{2n}) = r^{2n+2}$ is the required pedal equation.

---

84. *Find the length of the perpendicular from the pole to the tangent at the point $(a, \pi/2)$ on the curve $r = a(1-\cos\theta)$*

>> We have $r = a(1-\cos\theta)$

$\Rightarrow$ $\log r = \log a + \log(1-\cos\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$$

ie., $\cot\phi = \cot(\theta/2) \Rightarrow \phi = \theta/2$

Length of the perpendicular $p = r\sin\phi$

ie., $p = r\sin(\theta/2)$

Substituting $(r, \theta) = (a, \pi/2)$ we get $p = a\sin(\pi/4)$

Thus $p = a/\sqrt{2}$

---

85. [illegible] the tangent at the point [illegible]

>> We have $r = a\sec^2(\theta/2)$

$\Rightarrow \quad \log r = \log a + 2\log\sec(\theta/2)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2 \cdot \sec(\theta/2)\tan(\theta/2) \cdot 1/2}{\sec(\theta/2)} = \tan(\theta/2)$$

$\therefore \quad \cot\phi = \cot(\pi/2 - \theta/2) \Rightarrow \phi = \pi/2 - \theta/2$

Length of the perpendicular $p = r\sin\phi$

ie., $p = r\sin(\pi/2 - \theta/2)$ or $p = r\cos(\theta/2)$

We have at $\theta = \pi/3$, $r = a\sec^2(\pi/6) = 4a/3$

$$\therefore \quad p = \frac{4a}{3}\cos(\pi/6) = \frac{4a}{3} \cdot \frac{\sqrt{3}}{2} = \frac{2a}{\sqrt{3}}$$

Hence the length of the perpendicular $p = 2a/\sqrt{3}$

EXERCISES

Find the angle between the radius vector and the tangent for the following curves.

1. $r\sec^2(\theta/2) = 2a$
2. $r = a\,\mathrm{cosec}^2(\theta/2)$
3. $r^2 = a^2(\cos 2\theta + \sin 2\theta)$
4. $r^n\,\mathrm{cosec}\, n\theta = a^n$

Find the slopes of the tangents for the following curves at the indicated points.

5. $r^2 = a^2\sin 2\theta$ at $\theta = \pi/12$
6. $r\,\mathrm{cosec}\, 2\theta = a$ at $\theta = \pi/4$
7. $r = a\sin 3\theta$ at the pole
8. $r\sec^2(\theta/2) = 4$ at $\theta = \pi/2$

Show that the following pairs of curves intersect each other orthogonally.

9. $r\sec^2(\theta/2) = a$ and $r\,\mathrm{cosec}^2(\theta/2) = b$
10. $r^n\cos n\theta = a^n$ and $r^n\sin n\theta = b^n$

11. $2a/r = 1 + \cos\theta$ and $2a/r = 1 - \cos\theta$

12. $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$

Find the angle of intersection for the following pairs of curves.

13. $r = a\cos\theta$ and $r = a/2$

14. $r^n = a^n (\sin n\theta + \cos n\theta)$ and $r^n = a^n \sin n\theta$

15. $r^2 \cos(2\theta + \alpha) = a^2$ and $r^2 \cos(2\theta + \beta) = b^2$

16. $r^2 = a^2 \cos 2\theta + b^2$ and $r = b$

Obtain the pedal equation of the following curves.

17. $r^2 \cos 2\theta = a^2$

18. $r = 2a/1 + \cos\theta$

19. $r = a \operatorname{sech} n\theta$

20. $r = a + b\cos\theta$

21. $r^2 = a^2 \sin 2\theta + b^2 \cos 2\theta$

22. $r = a\sin 3\theta$

23. $r^n \sec n\theta = a^n$

24. Show that for the curve $r\sin^2(\theta/2) = a$ the length of the perpendicular from the pole to the tangent at the point $(2a\ \pi/2)$ on the curve is equal to $a\sqrt{2}$.

25. Show that the length of the perpendicular from the pole to the tangent at the point $\theta = \pi/6$ on the curve $r^2\cos 2\theta = a^2$ is equal to $a/\sqrt{2}$.

ANSWERS

1. $\pi/2 + \theta/2$
2. $-\theta/2$
3. $\pi/4 + 2\theta$
4. $n\theta$
5. 1
6. -1
7. 0
8. 1
13. $\pi/3$
14. $\pi/4$
15. $\alpha - \beta$
16. $\tan^{-1}(a^2/b^2)$
17. $pr = a^2$
18. $p^2 = ar$
19. $\dfrac{1}{p^2} = \dfrac{n^2+1}{r^2} - \dfrac{n^2}{a^2}$
20. $p^2[2ar + b^2 - a^2] = r^4$
21. $r^6 = p^2(a^4 + b^4)$
22. $p^2(9a^2 - 8r^2) = r^4$
23. $pa^n = r^{n+1}$

2.3 Derivative of Arc length

2.31 Introduction

An arc of a curve is a part of it and we are familiar with the various form of curves : Cartesian form $[\, y = f(x)\,]$, Parametric form $[\, x = x(t),\ y = y(t)\,]$, Polar form $[\, r = f(\theta)\,]$, pedal form $[\, f(r, p) = c\,]$. Length of an arc of a curve is usually denoted by 's' and several results connected with the derivative of arc length 's' can be established from the basic definition. We **assume** these well established results as it is an **essential pre-requisite** for the study of the following topic called **Radius of Curvature**.

2.32 Formulae connected with the derivative of arc length.

1. Cartesian curve : $y = f(x)$

(i) $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ (ii) $\sin\psi = \dfrac{dy}{ds}$ (iii) $\cos\psi = \dfrac{dx}{ds}$ (iv) $\tan\psi = \dfrac{dy}{dx}$

ψ being the angle made by the tangent at $P\,(x, y)$ on the curve with the X – axis.

2. **Parametric curve :** $x = x(t), y = y(t)$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

3. **Polar curve :** $r = f(\theta)$

(i) $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$ (ii) $\sin\phi = r\,\dfrac{d\theta}{ds}$ (iii) $\cos\phi = \dfrac{dr}{ds}$

ϕ being the angle made by the radius vector and the tangent at $P(r, \theta)$ on the polar curve $r = f(\theta)$.

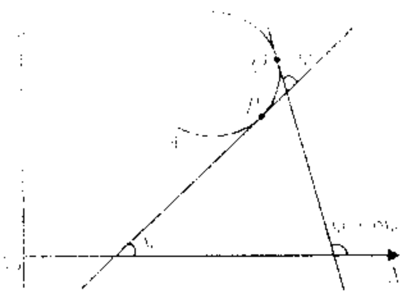
2.4 Radius of curvature

2.41 Introduction

If we traverse in a ghat section (hilly region) where the road is not straight, we often see caution boards "sharp bend ahead", "hairpin bend ahead" etc. which gives an indication of the difference in the amount of bending of a road at various points which is nothing but *curvature* at various points and we discuss the same in a mathematical way. This aspect is discussed for cartesian, parametric, polar and pedal form of curves.

2.42 Curvature and Radius of Curvature

Definition



Consider a curve in the XOY plane and let A be a fixed point on it. Let P and Q be two neighbouring points on the curve such that,

$\overset{\frown}{AP} = s$ and $\overset{\frown}{AQ} = s+\delta s$ so that $\overset{\frown}{PQ} = \delta s$.

Let ψ and $\psi+\delta\psi$ respectively be the angles made by the tangents at P and Q with the X - axis. The angle $\delta\psi$ between the tangents is called the bending of the curve which depends on δs. $\delta\psi/\delta s$ is called as the *mean curvature* of the arc PQ. Also the amount of bending of the curve at P is called as the ***curvature*** of the curve at P and is defined mathematically as

$$\lim_{\substack{\delta s \to 0 \\ (Q \to P)}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} \text{ be denoted by } K$$

ie., ***Curvature*** $= K = \dfrac{d\psi}{ds}$. Further if $K \neq 0$, the reciprocal of the curvature is called as the *radius of curvature* and is denoted by ρ.

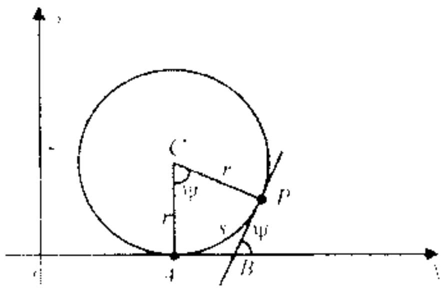
ie., ***Radius of curvature*** $= \rho = \dfrac{1}{K} = \dfrac{ds}{d\psi}$

Note :

1. *As it is obvious that ψ depends on s, the relationship between these is called as the intrinsic equation and (s, ψ) are called the intrinsic coordinates of the point P*
2. *We always take the sign of K and ρ to be positive.*

Remark : *Curvature being the amount of bending is obviously zero for a straight line at all the points on it. It is easy to visualize that the circle has an uniform bending and hence the curvature of a circle is a constant which will be established mathematically.*

A Question Format : *Define curvature and prove that the curvature of a circle is a constant.*



[Definition given already]

Consider a circle of radius r having centre at the point C. Let A be a fixed point on the circle and $P(x, y)$ be any point on the circle such that $\overset{\frown}{AP} = s$. Let ψ be the angle made by the tangent at P with the X-axis at the point B (*interior angle being* $\pi - \psi$). Clearly $CA = CP = r =$ radius.

We have from the quadrilateral $CABP$, $\hat{C} + \hat{A} + \hat{B} + \hat{P} = 2\pi$

ie., $\quad \hat{C} + \pi/2 + (\pi - \psi) + \pi/2 = 2\pi \quad \therefore \; A\hat{C}P = \psi$

We have a known result

$$s = r\psi \quad \text{or} \quad \psi = \frac{s}{r} \quad \therefore \; \frac{d\psi}{ds} = \frac{1}{r} = \text{constant.}$$

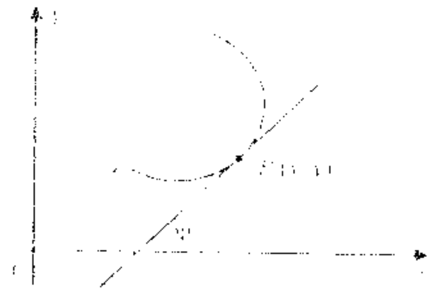
Thus the curvature $K = 1/r =$ **constant.**

This proves that the curvature of a circle at any point on it is a constant and is equal to the reciprocal of the radius.

We now proceed to derive expressions for the radius of curvature in respect of cartesian, parametric, polar and pedal form of curves.

2.43 An expression for the radius of curvature in the case of a cartesian curve

Let $y = f(x)$ be the equation of the cartesian curve and A be a fixed point on it. Let $P(x, y)$ be a point on the curve such that $\overset{\frown}{AP} = s$. Let ψ be the angle made by the tangent at P with the x-axis.



Then we know that $\tan\psi = \dfrac{dy}{dx}$

Differentiating w.r.t s we have,

$$\frac{d}{ds}(\tan\psi) = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

ie., $$\sec^2\psi \frac{d\psi}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$$

But $\dfrac{dx}{ds} = \cos\psi$ and by the definition $\dfrac{d\psi}{ds} = \dfrac{1}{\rho}$

$$\therefore \quad \sec^2\psi \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2}\cos\psi \quad \text{or} \quad \sec^3\psi = \rho\,\frac{d^2y}{dx^2}$$

Hence $\rho = \sec^3\psi \Big/ \dfrac{d^2y}{dx^2}$

ie., $$\rho = \frac{(\sec^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

Denoting $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$ we have,

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

Note : *Sometimes* y_1 *at some point on the curve becomes infinity (ie., when the tangent is perpendicular to the x-axis,* $\tan\psi = \tan 90^\circ = \infty$*) in which case we cannot apply the formula for* ρ *in the above form. In such a case we have to use the formula in the alternative form,*

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy} \text{ and } x_2 = \frac{d^2x}{dy^2}$$

[**Note** : *The expression for* ρ *in the case of* $y = f(x)$ *has to be established first.*]

We have for a cartesian curve $y = f(x)$,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad \ldots (1)$$

We shall express y_1 and y_2 in terms of the parameter t.

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\,\frac{dx}{dt} = \frac{y'}{x'} \text{ where } y' = \frac{dy}{dt},\ x' = \frac{dx}{dt}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx} = \frac{x'y''-y'x''}{(x')^2}\cdot\frac{1}{x'}$$

ie., $$y_2 = \frac{x'y''-y'x''}{(x')^3} \text{ where } y'' = \frac{d^2y}{dt^2} \text{ and } x'' = \frac{d^2x}{dt^2}$$

Substituting in (1) we get,

$$\rho = \frac{\left\{1+(y'/x')^2\right\}^{3/2}}{x'y''-y'x''}\cdot(x')^3$$

$$= \frac{\left\{(x')^2+(y')^2\right\}^{3/2}}{\left\{(x')^2\right\}^{3/2}\cdot(x'y''-y'x'')}\cdot(x')^3$$

$$= \frac{\left\{(x')^2+(y')^2\right\}^{3/2}}{(x')^3(x'y''-y'x'')}\cdot(x')^3$$

Thus $\boxed{\rho = \frac{\left\{(x')^2+(y')^2\right\}^{3/2}}{x'y''-y'x''}}$

In an alternative notation with the same meaning the above expression is also put in the form

$$\rho = \frac{\left\{(\dot{x})^2+(\dot{y})^2\right\}^{3/2}}{\dot{x}\,\ddot{y}-\dot{y}\,\ddot{x}}$$

We prefer to use the cartesian formula itself for finding ρ *in the case of parametric curves also as the work will be relatively easy.*

>> $s = a\log\tan(\pi/4+\psi/2)$ and we have $\rho = \frac{ds}{d\psi}$

Differentiating w.r.t ψ we have,

$$\frac{ds}{d\psi} = a\cdot\frac{1}{\tan(\pi/4+\psi/2)}\cdot\sec^2(\pi/4+\psi/2)\cdot\frac{1}{2}$$

$$= \frac{a}{2}\cdot\frac{\cos(\pi/4+\psi/2)}{\sin(\pi/4+\psi/2)}\cdot\frac{1}{\cos^2(\pi/4+\psi/2)}$$

$$= \frac{a}{2\sin(\pi/4+\psi/2)\cos(\pi/4+\psi/2)}$$ But $2\sin\theta\cos\theta = \sin 2\theta$

$$\therefore \quad \frac{ds}{d\psi} = \frac{a}{\sin[2(\pi/4+\psi/2)]} = \frac{a}{\sin(\pi/2+\psi)} = \frac{a}{\cos\psi} = a\sec\psi$$

Thus $\rho = a\sec\psi$

87. *Show that the radius of curvature for the catenary of uniform strength $y = a \log \sec(x/a)$ is $a \sec(x/a)$*

$$\text{>> We have } \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Consider $y = a \log \sec(x/a)$

$$\therefore \quad \frac{dy}{dx} = y_1 = \frac{a}{\sec(x/a)} \cdot \sec(x/a)\tan(x/a) \cdot \frac{1}{a}$$

ie., $y_1 = \tan(x/a)$. Also $y_2 = \dfrac{1}{a}\sec^2(x/a)$

$$\text{Hence } \rho = \frac{[1+\tan^2(x/a)]^{3/2} \cdot a}{\sec^2(x/a)} = \frac{a[\sec^2(x/a)]^{3/2}}{\sec^2(x/a)}$$

$$\text{ie., } \quad \rho = \frac{a\sec^3(x/a)}{\sec^2(x/a)} = a\sec(x/a)$$

Thus $\rho = a\sec(x/a)$

---

88. *Show that for the catenary [illegible] radius of curvature is equal to y^2/c which is also equal to the length of the normal intercepted between the curve and the x-axis.*

$$\text{>> We have } \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$y = c\cosh(x/c)$ by data

$$\therefore \quad y_1 = c \cdot \sinh(x/c) \cdot \frac{1}{c} = \sinh(x/c) \; ; \; y_2 = \frac{1}{c}\cosh(x/c)$$

$$\text{Hence } \quad \rho = \frac{[1+\sinh^2(x/c)]^{3/2}\, c}{\cosh(x/c)} = \frac{c[\cosh^2(x/c)]^{3/2}}{\cosh(x/c)}$$

$$\text{ie., } \quad \rho = \frac{c\cosh^3(x/c)}{\cosh(x/c)} = c\cosh^2(x/c)$$

But $y/c = \cosh(x/c)$ and hence $\rho = c \cdot (y^2/c^2) = y^2/c$

Also we know that the length of the normal (l) is $y\sqrt{1+y_1^2}$

$$\therefore \quad l = c\cosh(x/c)\sqrt{1+\sinh^2(x/c)} = c\cosh^2(x/c) = y^2/c$$

This proves the required result

89. *Find the radius of curvature for the curve* $y = ax^2 + bx + c$ *at* $x = \frac{1}{2a}\left[\sqrt{a^2-1} - b\right]$

\>> $y = ax^2 + bx + c$, by data.

$\therefore$ $y_1 = 2ax + b,\ y_2 = 2a$

At the given point, $y_1 = 2a \cdot \frac{1}{2a}\left[\sqrt{a^2-1} - b\right] + b = \sqrt{a^2-1}$ and $y_2 = 2a$ itself.

We have, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{[1+(a^2-1)]^{3/2}}{2a} = \frac{(a^2)^{3/2}}{2a} = \frac{a^2}{2}$$

Thus $\rho = a^2/2$

---

90. *Find the radius of curvature for the Folium of De Cartes* $x^3 + y^3 = 3axy$ *at the point* $(3a/2, 3a/2)$ *on it.*

\>> $x^3 + y^3 = 3axy$, by data.

Differentiating w.r.t. x we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

ie., $$3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2) \quad \therefore \frac{dy}{dx} = y_1 = \frac{ay - x^2}{y^2 - ax}$$

At $(3a/2, 3a/2)$, $y_1 = \dfrac{3a^2/2 - 9a^2/4}{9a^2/4 - 3a^2/2} = -1$

Next $$\frac{d^2y}{dx^2} = y_2 = \frac{(y^2-ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

At $(3a/2, 3a/2)$ we note that, $y^2 - ax = 9a^2/4 - 3a^2/2 = 3a^2/4$ and $ay - x^2 = 3a^2/2 - 9a^2/4 = -3a^2/4$.

Hence at $(3a/2, 3a/2)$,

$$y_2 = \frac{(3a^2/4)(-a-3a) - (-3a^2/4)(-3a-a)}{(3a^2/4)^2}$$

ie., $$y_2 = \frac{-3a^3 - 3a^3}{9a^4/16} = \frac{16(-6a^3)}{9a^4} = \frac{-32}{3a}$$

We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Hence $\rho = \dfrac{(1+1)^{3/2}}{-32/3a} = \dfrac{2\sqrt{2}\cdot 3a}{-32} = \dfrac{-3\sqrt{2}\,a}{16} = \dfrac{-3a}{8\sqrt{2}}$

Thus $|\rho| = 3a/8\sqrt{2}$

\>> If the curve meets the x-axis then $y = 0$.

$\therefore \quad \dfrac{4a^2(2a-x)}{x} = 0 \Rightarrow 4a^2(2a-x) = 0 \quad \therefore\ x = 2a$

Thus $(2a, 0)$ is the point on the curve at which we have to find ρ.
The given equation can be put in the form

$$y^2 = \frac{8a^3}{x} - 4a^2$$

Differentiating w.r.t. x we have $2yy_1 = -\dfrac{8a^3}{x^2}$ or $y_1 = \dfrac{-4a^3}{x^2 y}$

At $(2a, 0)$ y_1 becomes infinity and hence we have to consider dx/dy.

Let $\quad x_1 = \dfrac{dx}{dy} = \dfrac{-x^2 y}{4a^3}$ and $x_1 = 0$ at $(2a, 0)$

Now $\quad x_2 = \dfrac{d^2x}{dy^2} = \dfrac{-1}{4a^3}\left[x^2\cdot 1 + y\cdot 2x\,x_1\right]$

$\therefore \quad$ at $(2a, 0)$: $x_2 = -4a^2/4a^3 = -1/a$

We have $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$$= \frac{(1+0)^{3/2}}{-1/a} = -a$$

Thus $\quad |\rho| = a$

\>> Consider $x^2 y = a(x^2+y^2)$ and differentiate w.r.t. x

$\therefore \quad x^2 y_1 + 2xy = 2ax + 2ayy_1$

ie., $y_1(x^2-2ay) = 2ax-2xy$

or $y_1 = \dfrac{2ax-2xy}{x^2-2ay}$; At $(-2a, 2a)$, y_1 is infinity.

Hence $x_1 = \dfrac{dx}{dy} = \dfrac{1}{y_1} = \dfrac{x^2-2ay}{2ax-2xy}$ and at $(-2a,\ 2a)$ we have $x_1 = 0$

Also $\dfrac{d^2x}{dy^2} = \dfrac{(2ax-2xy)(2xx_1-2a)-(x^2-2ay)(2ax_1-2x-2x_1y)}{(2ax-2xy)^2}$

We note that at $(-2a, 2a)$

$$(2ax-2xy) = 4a^2 \text{ and } (x^2-2ay) = 0$$

$$\therefore \quad (x_2)_{(-2a,\,2a)} = \frac{(4a^2)(-2a)}{16a^4} = \frac{-1}{2a}$$

We have, $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$$= \frac{(1)^{3/2}}{-1/2a} = -2a$$

Thus $|\rho| = 2a$

>> The equation of the line is $y = x$ and we shall find the point of intersection of this line with the curve $\sqrt{x}+\sqrt{y} = 4$.

This equation when $y = x$ becomes,

$$\sqrt{x}+\sqrt{x} = 4 \text{ or } 2\sqrt{x} = 4 \text{ or } \sqrt{x} = 2 \text{ or } x = 4$$

$\therefore$ the point of intersection is $(4,\ 4)$

Consider $\sqrt{x}+\sqrt{y} = 4$ and differentiate w.r.t. x

$$\therefore \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\,y_1 = 0 \text{ or } \frac{y_1}{\sqrt{y}} = \frac{-1}{\sqrt{x}}$$

ie., $y_1 = -\sqrt{y}/\sqrt{x}$. At $(4, 4)$ we get $y_1 = -1$

Now $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{\sqrt{x}(-1/2\sqrt{y}\cdot y_1)-(-\sqrt{y})(1/2\sqrt{x})}{x}$

$\therefore$ at $(4, 4)$, $y_2 = \dfrac{1/2 + 1/2}{4} = \dfrac{1}{4}$

We have, $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{(2)^{3/2}}{1/4} = 4 \cdot 2\sqrt{2} = 8\sqrt{2}$$

Thus $\rho = 8\sqrt{2}$

[illegible]

>> $y = \dfrac{ax}{a + x}$, by data.

$\therefore$ $y_1 = \dfrac{(a + x)a - ax \cdot 1}{(a + x)^2} = \dfrac{a^2}{(a + x)^2}$

Also $y_2 = \dfrac{-2a^2}{(a + x)^3}$

We have $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

Hence $\rho = \dfrac{\left[1 + \dfrac{a^4}{(a + x)^4}\right]^{3/2} \cdot (a + x)^3}{-2a^2}$

$$= \frac{[(a + x)^4 + a^4]^{3/2} \cdot (a + x)^3}{-2a^2 \left\{(a + x)^4\right\}^{3/2}}$$

$$= \frac{[(a + x)^4 + a^4]^{3/2} \cdot (a + x)^3}{-2a^2 (a + x)^6}$$

or $-2\rho = \dfrac{[(a + x)^4 + a^4]^{3/2}}{a^2 (a + x)^3}$

$\Rightarrow$ $(-2\rho)^{2/3} = \dfrac{(a + x)^4 + a^4}{a^{4/3} (a + x)^2}$; We note that $(-2)^{2/3} = 2^{2/3}$

$\therefore$ $(2\rho)^{2/3} = \dfrac{1}{a^{4/3}} \left\{ (a + x)^2 + \left(\dfrac{a^2}{a + x}\right)^2 \right\}$

But $\quad a+x = \dfrac{ax}{y}$ by data.

$$\therefore \quad (2\rho)^{2/3} = \frac{1}{a^{4/3}}\left\{\frac{a^2x^2}{y^2}+\frac{a^2y^2}{x^2}\right\}$$

ie., $\quad (2\rho)^{2/3} = a^{2/3}\left\{\left(\dfrac{x}{y}\right)^2+\left(\dfrac{y}{x}\right)^2\right\}$

Thus $\quad (2\rho/a)^{2/3} = (x/y)^2+(y/x)^2$

---

95. *Find the radius of curvature of the curve* $x = a\log(\sec t+\tan t)$, $y = a\sec t$

\>> $\quad x = a\log(\sec t+\tan t)$

$$\frac{dx}{dt} = \frac{a}{\sec t+\tan t}\cdot \sec t\tan t+\sec^2 t = \frac{a\sec t(\sec t+\tan t)}{\sec t+\tan t}$$

$$\therefore \quad \frac{dx}{dt} = a\sec t$$

Also $\quad y = a\sec t\;$ gives $\;\dfrac{dy}{dt} = a\sec t\tan t$

Now, $\; y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt}\quad\dfrac{dx}{dt} = \dfrac{a\sec t\tan t}{a\sec t} = \tan t$

Differentiating w.r.t. x we get, $\; y_2 = \sec^2 t\,\dfrac{dt}{dx}$

$$\therefore \quad y_2 = \sec^2 t\cdot\frac{1}{a\sec t} = \frac{\sec t}{a}$$

We have $\; \rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}a}{\sec t} = \frac{a\sec^3 t}{\sec t}$$

Thus $\quad \rho = a\sec^2 t$

---

96. *Show that the radius of curvature at any point* θ *on the cycloid* $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ *is* $4a\cos(\theta/2)$

\>> $\quad x = a(\theta+\sin\theta)\quad ; \quad y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1+\cos\theta)\quad ; \quad \frac{dy}{d\theta} = a\sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}\quad\frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$\therefore \quad y_1 = \tan(\theta/2)$

Differentiating w.r.t. x we get,

$$y_2 = \sec^2(\theta/2) \cdot \frac{1}{2} \cdot \frac{d\theta}{dx}$$

$$= \sec^2(\theta/2) \cdot \frac{1}{2} \cdot \frac{1}{a(1+\cos\theta)} = \frac{\sec^2(\theta/2)}{4a\cos^2(\theta/2)}$$

$$\therefore \quad y_2 = \frac{1}{4a}\sec^4(\theta/2)$$

We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+\tan^2(\theta/2)]^{3/2} \cdot 4a}{\sec^4(\theta/2)}$$

$$= \frac{[\sec^2(\theta/2)]^{3/2} \cdot 4a}{\sec^4(\theta/2)} = \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)}$$

Thus $\quad \boldsymbol{\rho = 4a\cos(\theta/2)}$

>> For the given curve we have

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)} \cdot \sec^2(t/2) \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\cos(t/2)\sin(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{-\sin^2 t + 1}{\sin t}\right] = a \cdot \frac{\cos^2 t}{\sin t}$$

ie., $\quad \dfrac{dx}{dt} = a\cos^2 t \,\operatorname{cosec} t$

Also $\quad \dfrac{dy}{dt} = a\cos t$

Now, $\quad y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt} = \dfrac{a\cos t}{a\cos^2 t\,\operatorname{cosec} t} = \tan t$

Hence $y_2 = \sec^2 t \dfrac{dt}{dx} = \dfrac{\sec^2 t}{a\cos^2 t \ \operatorname{cosec} t} = \dfrac{\sec^4 t \ \sin t}{a}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2} \cdot a}{\sec^4 t \ \sin t} = \frac{a\sec^3 t}{\sec^4 t \sin t}$$

Thus $\quad \boldsymbol{\rho = a \cot t}$

---

98. [illegible] the radius of curvature [illegible] $a\sin^3\theta$ at $\theta = \pi/4$

>> $\quad x = a\cos^3\theta \quad ; \quad y = a\sin^3\theta$

$\therefore \quad \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad ; \quad \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Now $\quad y_1 = \dfrac{dy}{dx} = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta} = \dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$

Hence $y_2 = -\sec^2\theta \cdot \dfrac{d\theta}{dx} = \dfrac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \dfrac{\sec^4\theta \operatorname{cosec}\theta}{3a}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2\theta)^{3/2} \cdot 3a}{\sec^4\theta \ \operatorname{cosec}\theta} = \frac{3a\sec^3\theta}{\sec^4\theta\operatorname{cosec}\theta} = 3a\cos\theta \ \sin\theta$$

Thus at $\theta = \pi/4, \quad \boldsymbol{\rho = 3a/2}$

---

99. *Show that the radius of curvature of the curve* $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ *is 'at'.*

>> $\quad x = a(\cos t + t\sin t) \quad ; \quad y = a(\sin t - t\cos t)$

$\dfrac{dx}{dt} = a(-\sin t + t\cos t + \sin t) \ ; \quad \dfrac{dy}{dt} = a(\cos t + t\sin t - \cos t)$

$\therefore \quad \dfrac{dx}{dt} = at\cos t$ and $\dfrac{dy}{dt} = at\sin t$

Now, $\quad y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{a\,t\sin t}{a\,t\cos t} = \tan t$

Hence $y_2 = \sec^2 t \dfrac{dt}{dx} = \dfrac{\sec^2 t}{at\cos t} = \dfrac{\sec^3 t}{at}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}}{\sec^3 t}\cdot at = \frac{\sec^3 t}{\sec^3 t}\cdot at$$

Thus $\rho = at$

100. *If ρ be the radius of curvature at any point $P(x, y)$ on the parabola $y^2 = 4ax$ show that ρ^2 varies as $(SP)^3$ where S is the focus of the parabola.*

>> Consider $y^2 = 4ax$ and differentiate w.r.t. x

$\therefore \quad 2yy_1 = 4a$ or $y_1 = 2a/y$

Further $y_2 = \dfrac{-2a}{y^2}\cdot y_1 = \dfrac{-4a^2}{y^3}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{\{1+(4a^2/y^2)\}^{3/2}}{-4a^2/y^3} = \frac{y^3\{(y^2+4a^2)/y^2\}^{3/2}}{-4a^2}$$

$$= \frac{y^3}{-4a^2}\cdot\frac{(y^2+4a^2)^{3/2}}{(y^2)^{3/2}} = \frac{(y^2+4a^2)^{3/2}}{-4a^2}$$

ie., $\quad \rho = \dfrac{(4ax+4a^2)^{3/2}}{-4a^2} = \dfrac{(4a)^{3/2}(x+a)^{3/2}}{-4a^2}$

By squaring we have,

$$\rho^2 = \frac{(4a)^3(x+a)^3}{16a^4} = \frac{64a^3(x+a)^3}{16a^4}$$

ie., $\quad \rho^2 = \dfrac{4}{a}(x+a)^3 \qquad \ldots(1)$

The co-ordinates of the focus of the parabola is $S = (a, 0)$ and we have $P = (x, y)$

$\therefore \quad SP = \sqrt{(x-a)^2+(y-0)^2}$ by the distance formula.

$$= \sqrt{x^2-2ax+a^2+y^2} = \sqrt{x^2-2ax+a^2+4ax}$$

$$= \sqrt{x^2 + 2ax + a^2} = \sqrt{(x+a)^2} = (x+a)$$

Hence $SP = (x+a)$ and using this result in (1) we have,

$$\rho^2 = \frac{4}{a}(SP)^3$$

That is, $\rho^2 = \text{const.}\,(SP)^3$

Thus $\rho^2 \propto (SP)^3$

---

101. Prove that for the ellipse $x^2/a^2 + y^2/b^2 = 1$, the radius of curvature is equal to a^2b^2/p^3 where p is the length of the perpendicular from the centre of the ellipse upon the tangent at (x, y). Hence deduce that ρ at the end of the major axis is equal to the semi latus rectum.

>> The parametric equations of the ellipse are $x = a\cos\theta$, $y = b\sin\theta$ and we prefer to apply the parametric formula for finding ρ.

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \quad \text{for a parametric curve.}$$

$$\dot{x} = \frac{dx}{d\theta} = -a\sin\theta \quad ; \quad \dot{y} = \frac{dy}{d\theta} = b\cos\theta$$

$$\ddot{x} = \frac{d^2x}{d\theta^2} = -a\cos\theta \quad ; \quad \ddot{y} = \frac{d^2y}{d\theta^2} = -b\sin\theta$$

$$\therefore \quad \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab(\sin^2\theta + \cos^2\theta)}$$

ie.,

$$\rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

Further, equation of the tangent to the ellipse at $P\ (a\cos\theta, b\sin\theta)$ is given by $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. Also the length of the perpendicular from a point (x_1, y_1) upon a straight line $Ax + By + C = 0$ is given by the formula

$$p = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Hence the length of the perpendicular from the centre $O = (0, 0)$ of the ellipse upon the tangent $(\cos\theta/a)\,x + (\sin\theta/b)\,y - 1 = 0$ is given by

$$p = \frac{|0 + 0 - 1|}{\sqrt{\cos^2\theta/a^2 + \sin^2\theta/b^2}} = \frac{1}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)/a^2b^2}}$$

ie., $p = \dfrac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \Rightarrow p^3 = \dfrac{a^3 b^3}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}$

$\therefore \quad (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2} = a^3 b^3 / p^3$ ... (2)

Using (2) in (1) we get, $\rho = \dfrac{a^3 b^3 / p^3}{ab} = \dfrac{a^2 b^2}{p^3}$

Thus $\rho = a^2 b^2 / p^3$

Further, at the end of the major axis we have $(x, y) = (\pm a, 0)$

$\therefore \quad a \cos\theta = \pm a \Rightarrow \cos\theta = \pm 1$ or $\cos^2\theta = 1. \ \therefore \sin^2\theta = 0$

Hence $p = \dfrac{ab}{\sqrt{0 + b^2}}$ or $p = a$

Thus $\rho = a^2 b^2 / a^3 = b^2 / a$ being the length of the semi latus rectum.

This proves both the desired results.

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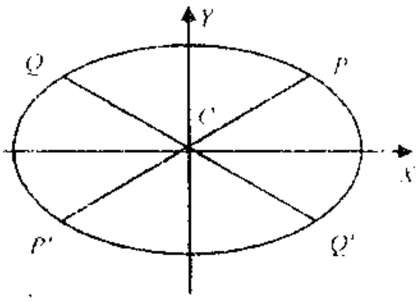
102. *If* ρ_1 *and* ρ_2 *be the radii of curvatures at the extremities of the two conjugate diameters of the ellipse, show that*

$\rho_1^{2/3} + \rho_2^{2/3} = (a^2 + b^2) / (ab)^{2/3}$

>> Let PCP' and QCQ' be the two conjugate diameters of the ellipse.



Noting that $x = a\cos\theta$ and $y = b\sin\theta$ represents the parametric equations of the ellipse and recollecting a property of the canjugate diameters with reference to the ecentric angle θ, we can write

$P = (a\cos\theta, b\sin\theta)$ and $Q = [a\cos(\pi/2 + \theta), b\sin(\pi/2 + \theta)]$

Let ρ_1 be the radius of curvature at P and ρ_2 be the radius of curvature at Q.

Hence $\rho_1 = \dfrac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab}$ ... (1)

[Refer the previous example]

Changing θ to $\pi/2 + \theta$ we have $\sin(\pi/2 + \theta) = \cos\theta$ and $\cos(\pi/2 + \theta) = -\sin\theta$. Thus we have from (1)

$$\rho_2 = \frac{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{3/2}}{ab}$$

Hence $\rho_1^{2/3} = \dfrac{a^2 \sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}}$; $\rho_2^{2/3} = \dfrac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}}$

$$\therefore \quad \rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2(\sin^2\theta + \cos^2\theta) + b^2(\cos^2\theta + \sin^2\theta)}{(ab)^{2/3}}$$

Thus $\rho_1^{2/3} + \rho_2^{2/3} = a^2 + b^2/(ab)^{2/3}$

2.15 An expression for the radius of curvature in the case of a polar curve $r = f(\theta)$

Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at $P(r, \theta)$.

Let ψ be the angle made by the tangent at P with the initial line.
Let A be a fixed point on the curve and let $\overset{\frown}{AP} = s$.

We have $\psi = \theta + \phi$

$$\therefore \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds} \quad ie., \ \frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\text{or} \quad \rho = \frac{\left(\dfrac{ds}{d\theta}\right)}{1 + \dfrac{d\phi}{d\theta}} \qquad \ldots (1)$$

We know that $\tan\phi = r\dfrac{d\theta}{dr} = r\Big/\left(\dfrac{dr}{d\theta}\right)$

ie., $\tan\phi = \dfrac{r}{r_1}$ where $r_1 = \dfrac{dr}{d\theta}$

Differentiating w.r.t θ we get,

$$\sec^2\phi \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r \cdot r_2}{r_1^2} \quad \text{where } r_2 = \frac{d^2 r}{d\theta^2}$$

$$\text{or} \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 \sec^2\phi} = \frac{r_1^2 - r r_2}{r_1^2(1 + \tan^2\phi)}$$

ie., $\dfrac{d\phi}{d\theta} = \dfrac{r_1^2 - r r_2}{r_1^2[1+(r^2/r_1^2)]} = \dfrac{r_1^2 - r r_2}{r_1^2 + r^2}$

Hence $1 + \dfrac{d\phi}{d\theta} = 1 + \dfrac{r_1^2 - r r_2}{r^2 + r_1^2} = \dfrac{r^2 + r_1^2 + r_1^2 - r r_2}{r^2 + r_1^2}$

ie., $1 + \dfrac{d\phi}{d\theta} = \dfrac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2}$ . . . (2)

Also, we know that $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r_1^2}$ . . . (3)

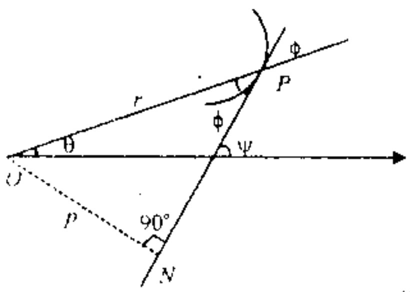
Using (2) and (3) in (1) we get

$$\rho = \sqrt{r^2 + r_1^2} \cdot \frac{(r^2 + r_1^2)}{r^2 + 2r_1^2 - r r_2}$$

Thus in the polar form, $\boxed{\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$

2.46 An expression for the radius of curvature in the case of a pedal curve

Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at P. Let ψ be the angle made by the tangent at P with the initial line. Draw $ON = p$, a perpendicular from the pole to the tangent.



We have from the Δ ONP, $\sin\phi = \dfrac{p}{r}$

ie., $p = r\sin\phi$ . . . (1)

Differentiating (1) w.r.t r we get,

$$\frac{dp}{dr} = r\cos\phi\,\frac{d\phi}{dr} + 1\cdot\sin\phi$$

But we know that, $\sin\phi = r\dfrac{d\theta}{ds}$ and $\cos\phi = \dfrac{dr}{ds}$

$$\therefore \quad \frac{dp}{dr} = r\frac{d\phi}{dr}\,\frac{dr}{ds} + r\,\frac{d\theta}{ds} = r\left[\frac{d\phi}{ds} + \frac{d\theta}{ds}\right] = r\frac{d}{ds}\,(\phi+\theta)$$

But $\phi + \theta = \psi$

$$\therefore \quad \frac{dp}{dr} = r\frac{d\psi}{ds} \quad \text{or} \quad \frac{ds}{d\psi} = r\frac{dr}{dp}$$

Thus $\boxed{\rho = r\dfrac{dr}{dp}}$

Note for problems

To find ρ for a polar curve $r = f(\theta)$, we have two options

(i) Applying the polar form of ρ by finding r_1 and r_2

(ii) Applying the pedal form of ρ by first finding the pedal equation of the curve as discussed already.

In the case of polar curves we prefer to take logarithms first and then differentiate w.r.t. θ.

WORKED PROBLEMS

103. *Show that for the equiangular spiral $r = a\,e^{\theta \cot \alpha}$ where a and α are constants, ρ/r is constant.*

>> $\quad r = ae^{\theta \cot \alpha}$

$\Rightarrow \quad \log r = \log a + \theta \cot \alpha \log e$. But $\log_e e = 1$

Differentiating w.r.t. θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \cdot \cot \alpha \quad ie., \quad \frac{dr}{d\theta} = r_1 = r\cot \alpha$$

Hence $\dfrac{d^2 r}{d\theta^2} = r_2 = r_1 \cot \alpha = (r\cot \alpha)\cot \alpha = r\cot^2 \alpha$

We have, $\quad \rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\therefore \quad \rho = \frac{(r^2 + r^2\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha} = \frac{(r^2)^{3/2}(\text{cosec}^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}$$

$$ie., \quad \rho = \frac{r^3\,\text{cosec}^3\alpha}{r^2\,\text{cosec}^2\alpha} = r\,\text{cosec}\,\alpha$$

Thus $\rho/r = \text{cosec}\,\alpha = $ **constant**.

Aliter : ***By applying the pedal form of ρ***

The pedal equation of the given curve is $p = r\sin \alpha$ [*Refer Example-80*]

Differentiating w.r.t p we get,

$$1 = \frac{dr}{dp}\sin\alpha \quad \therefore \quad \frac{dr}{dp} = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$$

Hence $\rho = r\,\dfrac{dr}{dp} = r\operatorname{cosec}\alpha$

Thus $\rho/r = \operatorname{cosec}\alpha$ = constant.

104. *Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ varies inversely as r^{n-1}*

>> $r^n = a^n \cos n\,\theta$

$\Rightarrow$ $n\log r = n\log a + \log(\cos n\,\theta)$

Differentiating w.r.t. θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n\sin n\,\theta}{\cos n\,\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\,\theta$$

$\therefore$ $r_1 = -r\tan n\,\theta$

Hence $r_2 = \dfrac{d^2 r}{d\theta^2} = -r_1\tan n\,\theta - n\,r\sec^2 n\,\theta$

We have $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\therefore \quad \rho = \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r(-r_1\tan n\theta - nr\sec^2 n\,\theta)}$$

$$= \frac{(r^2)^{3/2}(\sec^2 n\,\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\,\theta - r^2\tan^2 n\,\theta + nr^2\sec^2 n\,\theta}$$

$$= \frac{r^3\sec^3 n\,\theta}{r^2(1 + \tan^2 n\,\theta + n\sec^2 n\,\theta)}$$

$$= \frac{r\sec^3 n\,\theta}{\sec^2 n\,\theta\,(1+n)} = \frac{r\sec n\,\theta}{(1+n)}$$

Thus $\rho = \dfrac{r}{1+n}\sec n\,\theta$

But $a^n/r^n = \sec n\,\theta$ by data.

$\therefore \quad \rho = \dfrac{r}{1+n} \cdot \dfrac{a^n}{r^n} = \left[\dfrac{a^n}{1+n}\right] \dfrac{1}{r^{n-1}}$

ie., $\quad \rho = \text{const} \cdot \dfrac{1}{r^{n-1}}$

Thus $\rho \propto 1/r^{n-1}$

Aliter : ***By the pedal form of*** ρ

The pedal equation of the given curve is $p\,a^n = r^{n+1}$ *[Refer Example-75]*

Differentiating w.r.t. p, we get,

$$a^n = (n+1)\,r^n \frac{dr}{dp} \quad \therefore \quad \frac{dr}{dp} = \frac{a^n}{(n+1)\,r^n}$$

Hence $\rho = r \dfrac{dr}{dp} = r \cdot \dfrac{a^n}{(n+1)\,r^n} = \dfrac{a^n}{(n+1)} \cdot \dfrac{1}{r^{n-1}} = \text{const.} \dfrac{1}{r^{n-1}}$

Thus $\rho \propto 1/r^{n-1}$

---

105. *Show that for the curve $r(1-\cos\theta) = 2a$, ρ^2 varies as r^3*

\>> $\quad r(1-\cos\theta) = 2a$

$\Rightarrow \quad \log r + \log(1-\cos\theta) = \log 2a$

Differentiating w.r.t. θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \quad \text{or} \quad \frac{dr}{d\theta} = \frac{-r\sin\theta}{1-\cos\theta}$$

ie., $\quad \dfrac{dr}{d\theta} = \dfrac{-2r\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -r\cot(\theta/2)$

ie., $\quad r_1 = -r\cot(\theta/2)$

Hence $r_2 = -r \cdot \dfrac{-1}{2}\operatorname{cosec}^2(\theta/2) - r_1\cot(\theta/2)$

ie., $\quad r_2 = \dfrac{r}{2}\operatorname{cosec}^2(\theta/2) + r\cot^2(\theta/2)$

We have $\quad \rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

$$\therefore \quad \rho = \frac{\{r^2 + r^2 \cot^2(\theta/2)\}^{3/2}}{r^2 + 2r^2 \cot^2(\theta/2) - \frac{r^2}{2}\operatorname{cosec}^2(\theta/2) - r^2\cot^2(\theta/2)}$$

$$= \frac{(r^2)^{3/2}\{\operatorname{cosec}^2(\theta/2)\}^{3/2}}{r^2\left\{1 + \cot^2(\theta/2) - \frac{1}{2}\operatorname{cosec}^2(\theta/2)\right\}}$$

$$= \frac{r\operatorname{cosec}^3(\theta/2)}{1/2 \cdot \operatorname{cosec}^2(\theta/2)} = 2r\operatorname{cosec}(\theta/2)$$

$$\rho = 2r\operatorname{cosec}(\theta/2) \qquad \ldots (1)$$

But $r(1 - \cos\theta) = 2a$, by data.

ie., $r \cdot 2\sin^2(\theta/2) = 2a$ or $\sin^2(\theta/2) = a/r$

$\therefore$ $\operatorname{cosec}(\theta/2) = \sqrt{r/a}$ and hence (1) becomes

$$\rho = 2r \cdot \sqrt{r/a} = 2\,r^{3/2}/\sqrt{a}$$

Thus $\rho^2 = 4r^3/a = (4/a) \cdot r^3 \Rightarrow \rho^2 \propto r^3$

Aliter : ***By the pedal form of*** ρ

The pedal equation of the given curve is $p^2 = ar$ [*Refer Example*-73]

Differentiating w.r.t. p,

$$2p = a\frac{dr}{dp} \text{ or } \frac{dr}{dp} = \frac{2p}{a} = \frac{2\sqrt{ar}}{a} = \frac{2\sqrt{r}}{\sqrt{a}}$$

Hence $\rho = r\dfrac{dr}{dp} = r \cdot \dfrac{2\sqrt{r}}{\sqrt{a}}$

ie., $\rho = \dfrac{2}{\sqrt{a}}(r)^{3/2}$

Thus $\rho^2 = (4/a) \cdot r^3 \Rightarrow \rho^2 \propto r^3$

106. *Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole.*

>> $r = a\sin n\theta$

$\therefore$ $r_1 = an\cos n\theta,\ r_2 = -an^2\sin n\theta$

At the pole we have $\theta = 0$. When $\theta = 0$: $r = 0,\ r_1 = an,\ r_2 = 0$

We have $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\therefore \qquad \rho = \frac{(a^2 n^2)^{3/2}}{2a^2 n^2} = \frac{a^3 n^3}{2a^2 n^2} = \frac{a\,n}{2}$$

Thus $\boldsymbol{\rho = a n/2}$ at the pole.

107. *Show that at the point where the curve $r = a\theta$ intersects the curve $r = a/\theta$ their curvatures are in the ratio 3 : 1*

\>> Equating the R.H.S of the two given equations

$r = a\theta$ and $r = a/\theta$ we have,

$$a\theta = \frac{a}{\theta} \text{ or } \theta^2 = 1 \qquad \therefore\ \theta = \pm 1$$

Now $r = a\theta$, gives $r_1 = a,\ r_2 = 0$

At $\theta = +1,\ r = a,\ r_1 = a,\ r_2 = 0$ ... (1)

Also $r = a/\theta$ gives $r_1 = -a/\theta^2,\ r_2 = 2a/\theta^3$

At $\theta = +1,\ r = a,\ r_1 = -a,\ r_2 = 2a$ ... (2)

We have $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

From (1), $\rho = \dfrac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2} = \dfrac{(2a^2)^{3/2}}{3a^2} = \dfrac{2\sqrt{2}\,a}{3}$ ... (3)

From (2), $\rho = \dfrac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2 - 2a^2} = \dfrac{(2a^2)^{3/2}}{a^2} = \dfrac{2\sqrt{2}\,a}{1}$ ... (4)

Hence we have from (3) and (4) the ratio of the corresponding curvatures is given by

$$\frac{3/2\sqrt{2}\,a}{1/2\sqrt{2}\,a} = \frac{3}{1}$$

Thus, **the curvatures are in the ratio 3:1**

108. (a) *Show that for the curve $r = a(1 + \cos\theta)$, ρ^2/r is a constant*

(b) *If ρ_1 and ρ_2 be the radii of curvatures at the extremities of the polar chord of the* [illegible]

\>> (a) $r = a(1 + \cos\theta) \Rightarrow \log r = \log a + \log(1 + \cos\theta)$

Differentiating w.r.t. θ, we have,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

$\therefore \quad r_1 = -r\tan(\theta/2)$

Hence $r_2 = -\frac{r}{2}\sec^2(\theta/2) - r_1\tan(\theta/2)$

ie., $\quad r_2 = -\frac{r}{2}\sec^2(\theta/2) + r\tan^2(\theta/2)$

We have $\quad \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

$$\therefore \quad \rho = \frac{\left\{r^2+r^2\tan^2(\theta/2)\right\}^{3/2}}{r^2+2r^2\tan^2(\theta/2)+\frac{r^2}{2}\sec^2(\theta/2) - r^2\tan^2(\theta/2)}$$

$$= \frac{r^3\left\{\sec^2(\theta/2)\right\}^{3/2}}{r^2\left\{1+\tan^2(\theta/2)+\frac{1}{2}\sec^2(\theta/2)\right\}}$$

$$= \frac{r\sec^3(\theta/2)}{3/2\cdot\sec^2(\theta/2)} = \frac{2r}{3}\sec(\theta/2)$$

$$\rho = \frac{2r}{3}\sec(\theta/2) \qquad \ldots(1)$$

But $\quad r = a(1+\cos\theta) = a\cdot 2\cos^2(\theta/2)$

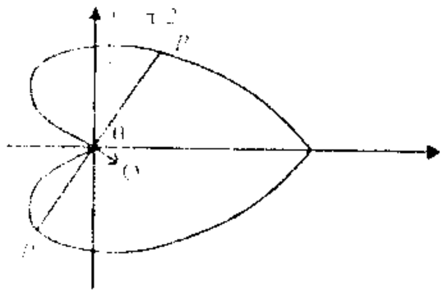
$\therefore \quad \sec^2(\theta/2) = \frac{2a}{r}$ or $\sec(\theta/2) = \frac{\sqrt{2a}}{\sqrt{r}}$

Hence (1) becomes $\rho = \frac{2r}{3}\frac{\sqrt{2a}}{\sqrt{r}}$ *ie.,* $\rho = \frac{2}{3}\sqrt{2ar}$

$\therefore \quad \rho^2 = \frac{4}{9}(2ar)$ or $\frac{\rho^2}{r} = \frac{8a}{9}$ = constant.

Thus ρ^2/r is a constant.

(b) Let POP' be the polar chord (*chord passing through the pole*) of the cardioide $r = a(1+\cos\theta)$. Let ρ_1 and ρ_2 be the radii of curvatures at the point P and P' corresponding to the vectorial angles θ and $(\pi+\theta)$ respectively.



We have already obtained

$$\rho_1 = \frac{2r}{3}\sec(\theta/2)$$

[*first part of this example*]

$$\therefore \quad \rho_1^2 = \frac{4r^2}{9}\sec^2(\theta/2)$$

But $\quad r = a(1+\cos\theta) = 2a\cos^2(\theta/2)$

$$\therefore \quad r^2 = 4a^2\cos^4(\theta/2)$$

Hence $\rho_1^2 = \frac{4}{9}\cdot 4a^2\cos^4(\theta/2)\sec^2(\theta/2)$

ie., $$\rho_1^2 = \frac{16a^2}{9}\cos^2(\theta/2) \qquad \ldots (2)$$

Now changing θ to $(\pi+\theta)$ we have from (2)

$$\rho_2^2 = \frac{16a^2}{9}\cos^2\left(\frac{\pi+\theta}{2}\right) = \frac{16a^2}{9}\cos^2(\pi/2+\theta/2)$$

ie., $$\rho_2^2 = \frac{16a^2}{9}\sin^2(\theta/2) \qquad \ldots (3)$$

Thus we have by adding (2) and (3)

$$\boldsymbol{\rho_1^2 + \rho_2^2 = 16a^2/9 = \text{constant.}}$$

---

109. *Show that the radius of curvature of the curve $pa^2 = r^3$ is $a^2/3r$*

Consider $pa^2 = r^3$ and differentiate w.r.t. p

$$\therefore \quad a^2 = 3r^2\frac{dr}{dp} \quad \text{or} \quad \frac{dr}{dp} = \frac{a^2}{3r^2}$$

We have $\rho = r\dfrac{dr}{dp}$

$$\therefore \quad \rho = r\cdot\frac{a^2}{3r^2} = \frac{a^2}{3r}$$

Thus $\quad \rho = a^2/3r$

---

110. *Show that for the ellipse in the pedal form* [illegible] *the radius of curvature at the point* (p, r) *is* $a^2 b^2/p^3$

>> Consider $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2 b^2}$

Differentiating w.r.t. p we get,

$$\frac{-2}{p^3} = \frac{-2r}{a^2 b^2}\frac{dr}{dp} \quad \therefore \quad \frac{dr}{dp} = \frac{a^2 b^2}{p^3 r}$$

We have, $\rho = r\,\dfrac{dr}{dp}$

$$\therefore \qquad \rho = r \cdot \frac{a^2 b^2}{p^3 r} = \frac{a^2 b^2}{p^3}$$

Thus $\rho = a^2 b^2/p^3$

Remark : *Referring to Example-101, we have obtained the same result starting from the equation of the ellipse in the parametric form.*

---

111. *Find the radius of curvature of the curve* $\theta =$ [illegible] $- \cos^{-1}(a/r)$ *at any point on it.*

>> Differentiating the given equation w.r.t. r we have,

$$\frac{d\theta}{dr} = \frac{1}{a}\cdot\frac{2r}{2\sqrt{r^2 - a^2}} - \left\{\frac{-1}{\sqrt{1-(a/r)^2}}\cdot\frac{-a}{r^2}\right\}$$

$$= \frac{r}{a\sqrt{r^2-a^2}} - \frac{r}{\sqrt{r^2-a^2}}\cdot\frac{a}{r^2}$$

$$= \frac{1}{\sqrt{r^2-a^2}}\left(\frac{r}{a} - \frac{a}{r}\right) = \frac{r^2 - a^2}{\sqrt{r^2-a^2}\cdot a r}$$

ie., $\qquad \dfrac{d\theta}{dr} = \dfrac{\sqrt{r^2 - a^2}}{ar}$ $\qquad$ . . . (1)

We prefer to find the pedal equation of the given curve and then apply the formula for ρ in the pedal form.

From (1) $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{r^2-a^2}}$ $\qquad$ *ie.,* $\cot\phi = \dfrac{a}{\sqrt{r^2-a^2}}$

Consider $p = r\sin\phi$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad ie., \ \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{a^2}{r^2 - a^2}\right]$$

$$ie., \quad \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{r^2}{r^2 - a^2}\right] \quad ie., \ \frac{1}{p} = \frac{1}{\sqrt{r^2 - a^2}}$$

$\therefore \quad p = \sqrt{r^2 - a^2}$ is the pedal equation of the curve.

Differentiating w.r.t. p we get,

$$1 = \frac{2r}{2\sqrt{r^2 - a^2}}\frac{dr}{dp} \quad ie., \ \sqrt{r^2 - a^2} = r\frac{dr}{dp} = \rho$$

Thus $\rho = \sqrt{r^2 - a^2}$

---

112. *Prove that* $\rho = p + \dfrac{d^2p}{d\psi^2}$ *with all the usual notations.*

>> We know that $p = r\sin\phi$

Now $\dfrac{dp}{d\psi} = \dfrac{dp}{dr}\dfrac{dr}{ds}\dfrac{ds}{d\psi}$ But $\dfrac{dr}{ds} = \cos\phi$ and $\dfrac{ds}{d\psi} = \rho = r\dfrac{dr}{dp}$

$$\therefore \quad \frac{dp}{d\psi} = \frac{dp}{dr}\cdot\cos\phi\cdot r\frac{dr}{dp} \quad \text{or} \quad \frac{dp}{d\psi} = r\cos\phi \qquad \ldots (2)$$

Squaring and adding (1) and (2) we get,

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2$$

Differentiating w.r.t p, we have,

$$2p + 2\frac{dp}{d\psi}\cdot\frac{d}{dp}\left(\frac{dp}{d\psi}\right) = 2r\frac{dr}{dp}$$

$$ie., \quad p + \frac{dp}{d\psi}\cdot\frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)\cdot\frac{d\psi}{dp} = r\frac{dr}{dp}$$

But $\rho = r\dfrac{dr}{dp}$ and $p + \dfrac{d^2p}{d\psi^2} = \rho$

Thus $\rho = p + \dfrac{d^2p}{d\psi^2}$

Note : *This form of expression for* ρ *is known as the tangential polar form.*

EXERCISES

Find the radius of curvature for the following curves [1 to 9]

1. $x^{2/3} + y^{2/3} = a^{2/3}$ at any point (x, y)
2. $xy^3 = a^4$ at the point (a, a)
3. $y^2 = x^3 + 8$ at the point $(-2, 0)$
4. $y = 4\sin x - \sin 2x$ at the point $(\pi/2, 4)$
5. $y = e^x$ at the point where the curve cuts the y-axis.
6. $y^2 = a^2(a - x)/x$ where the curve cuts the x-axis.
7. $x = a\log\sec\theta$, $y = a(\tan\theta - \theta)$
8. $x = a(t - \sin t)$, $y = a(1 - \cos t)$
9. $x = a\cos\theta$, $y = b\sin\theta$ at $(a/\sqrt{2}, b/\sqrt{2})$
10. Show that for the curve $r^n = a^n \sin n\theta$, ρ varies inversely as r^{n-1}
11. Show that for the curve $r^2 \sec 2\theta = a^2$, $\rho = a^2/3r$
12. Show that for the curve $r\cos^2(\theta/2) = a$, ρ^2 varies as r^3
13. Obtain the pedal equation of the curve $r = a(1 - \cos\theta)$ and hence show that $\rho = (2/3)\sqrt{2ar}$
14. Using the pedal formula for ρ, prove that $\rho = r^3/a^2$ for the curve $r^2 = a^2 \sec 2\theta$
15. Show that for the curve $p^2 = ar$, ρ^2 varies as r^3

ANSWERS

1. $3(axy)^{1/3}$	2. $5\sqrt{10}\,a/6$	3. 6
4. $5\sqrt{5}/4$	5. $2\sqrt{2}$	6. $a/2$
7. $a\tan\theta\sec\theta$	8. $4a\sin(t/2)$	9. $(a^2+b^2)^{3/2}/ab$